EURASIAN JOURNAL OF MATHEMATICAL
AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 5, Issue 4 (2017) 55 – 69

AN ALGORITHM FOR CALCULATING REFLECTED/TRANSMITTED POROELASTIC WAVES ACROSS ALL FREQUENCIES

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Abstract There is considered a layered heterogeneous poroelastic isotropic medium with physical parameters characterized by piecewise constant functions of the depth only. We derive a mathematical algorithm for calculating reflected/transmitted poroelastic waves across all temporal frequencies. To define the frequency effect we use the dynamic permeability expression proposed by Jonhson, Koplik and Dashen in 1987; in the time domain, this coefficient introduces order 1/2 shifted fractional time derivative involving a convolution product. The algorithm proposed is based on the formalism introduced by Ursin in 1983.

Key words: Stratified porous media, dynamic permeability, reflected/transmitted waves, frequency domain, explicit formulas.

AMS Mathematics Subject Classification: 86-06, 86A15.

1 Introduction

The theory of poroelasticity is essential in many applications, where porous materials are of interest, e.g., oil/gas exploration, petroleum engineering, rock physics, soil mechanics, and, in recent years, biomechanics. The attempt to capture the physical behavior of a porous material is quite old. First ideas can be even traced back to Leonhard Euler in the 18th century [1]. A historical treatment can be found in the review article by de Boer [2].

In 1956, Biot presented a theory of poroelasticity, which forms the basis of most investigations into elastic wave propagation in porous media containing a fluid [3, 4]. However, first works on poroelastodynamics are those of Frenkel [5] in 1944. Further work in the Russian scientific community based on this pioneering work is reviewed by Nikolaevskiy [6]. The connection of Frenkel's work to the Biot theory is presented by Pride and Garambois [7], where it is shown that both researchers have developed the same theory. The theory of Biot predicts two bulk compressional waves and one shear wave, which are dispersive and dissipative. The second bulk compressional wave, also known as the slow wave of Biot, was experimentally observed in a water-saturated porous solid by Plona [8] using an ultrasonic mode conversion technique.

Usually, the analysis of waves propagation in porous materials is restricted to seismic frequencies that are low enough so that the generation of viscous boundary layers in the pores of the rocks can be neglected. For example, for consolidated earth materials such as sandstones, the transition frequency at which viscous boundary layers first develop is typically greater than 100 kHz, so schemes that neglect this physics are valid for most seismic applications. By this reason, many methods have been developed for the low-frequency regime of the Biot equations, see [9]. However, in unconsolidated sediments, the transition frequency at which viscous boundary layers must be accounted for can be as small as 1 kHz or even less. Therefore, for many geophysical applications to unconsolidated or high permeability sediments, it is useful to have algorithms for solving Biot's equations across the entire band of frequencies. Furthermore, many laboratory experiments on porous materials are conducted at ultrasonic frequencies, in which case it is always necessary to account for the development of viscous boundary layers. In poroelastic theory, such pore-scale dynamics is allowed for in the time domain by using a dynamic permeability in a generalized Darcy's law. For the higher frequency range, Biot presented a formula for the dynamic permeability with two particular types of pore geometry: two-dimensional flow between parallel walls and three-dimensional flow in a circular duct [4]. Modeling this dynamic permeability behavior along with finding proper microstructural pore-space descriptors has received considerable attention in the literature [10]. Perhaps the most popular dynamic permeability model has been suggested by Johnson, Koplik and Dashen (JKD) [11]. They published a general expression for the dynamic permeability in the case of random pores, leading to the so-called Biot-JKD model. In this model, viscous stresses depend on the square root of the temporal frequency and only a non-dimensional physical parameter was involved.

Let us comment on some results in the literature concerning the Biot-JKD system. Up to now, the basic attention was addressed to the development of numerical methods and their applications. Carcione [12] presented a finite-differencing approach that allows for the dynamic permeability by approximating it as a sum of Zener relaxation functions. Hanyga and Lu [13] designed a numerical method based on the combination of the Fourier pseudo-spectral and predictor-corrector methods. Masson and Pride [14] used an explicit time-stepping finite-difference scheme for solving Biot's equations of poroelasticity across the entire band of frequencies. Blanc [15] proposed an explicit finite-difference scheme based on the diffusive representation of fractional derivatives when the convolution kernel is replaced by a finite number of memory variables that satisfy local-in-time ordinary differential equations. Li, Kouri and Chesnokov [16] proposed a new algorithm based on the rational expansion of dynamic permeability and the combination of the generalized phase-shift scheme and the pseudo-spectral method. Recently, Milani et al. [17] presented a finite-element technique to solve the one-dimensional Biot equations (with the dynamic permeability) in the space-frequency domain; there was considered the case of a medium composed of periodically distributed mesoscopic layers. A review of the various techniques and discussion of the numerical implementation aspects for application to seismic modeling and rock physics, as for instance the role of Biot's diffusion wave as a loss mechanism and interface waves in porous media, was done in [18].

We now comment on some theoretical results concerning existence, uniqueness, and continuous dependence of the solution to problem (1)–(3). To our knowledge the only paper about the subject was the paper of Lorenzi and Priimenko [19], where the authors studied well-posedness of an initial boundary-value problem for the Biot-JKD equations. As a result of the investigation, the authors proved a uniqueness and continuous dependence result for a generalization of the Biot-JKD equations related to a general bounded open set Ω in any spatial dimension $n = 1, 2, 3$, both on the unbounded time interval $(0, +\infty)$ and on the bounded time interval $(0, T)$; however, the existence question remains open.

In this paper we use a formalism introduced by Ursin [20], who showed how Maxwell's equations, the equations of acoustics and the equations of isotropic elasticity all have a similar mathematical structure in an appropriate way. We add the Biot-JKD equations (valid across of all temporal frequencies) to Ursin's list. We develop Ursin's formalism for the case of a stack of homogeneous layers, i.e., when the material parameters are piecewise functions of the depth only. In this case many quantities can be computed with explicit algebraic formulas. For the low-frequency range a similar algorithm was obtained by Azeredo [21].

We give now the plan of the paper. In Section 2 we formulate the Biot-JKD equations. In Section 3 we statement our problem. In Section 4 we construct the analytical expressions for the solution of the problem. Finally, two typical examples are considered in Section 5.

2 Biot-JKD equations

Using the dynamic permeability proposed in [11], we can rewrite the Biot system in the Biot-JKD form

$$\begin{aligned} &\rho\partial_t^2\boldsymbol{u} + \rho_f\partial_t^2\boldsymbol{w} = \nabla\cdot\boldsymbol{\tau} + \boldsymbol{f}, \\ &\rho_f\partial_t^2\boldsymbol{u} + \rho_w\partial_t^2\boldsymbol{w} + \frac{\eta}{\kappa_0}h * \left(\partial_t^2\boldsymbol{w} + \omega_c\partial_t\boldsymbol{w}\right) = -\nabla p + \boldsymbol{g}, \end{aligned} \tag{1}$$

with the constitutive laws (isotropic media case):

$$\begin{aligned} &\boldsymbol{\tau} = (\lambda\nabla\cdot\boldsymbol{u} + c\nabla\cdot\boldsymbol{w})\boldsymbol{I} + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T), \\ &p = -c\nabla\cdot\boldsymbol{u} - m\nabla\cdot\boldsymbol{w}, \end{aligned} \tag{2}$$

where the following notations were introduced:

$$\begin{aligned} &\rho_w = \rho_f F_e,\ \rho = \phi\rho_f + (1-\phi)\rho_s,\ \omega_c = \frac{2\pi f_c}{P}, \\ &h(t) = \frac{e^{-\omega_c t}}{\sqrt{\pi\omega_c t}},\ h * z(t,\cdot) = \int_0^t h(t-s)z(s,\cdot)ds. \end{aligned} \tag{3}$$

This model involves the following functions and physical parameters: the elastic stress tensor $\boldsymbol{\tau}$ and the acoustic pressure p, the relative displacement vector $\boldsymbol{w} = (w_1, w_2, w_3)^T$ of the fluid phase, the volume density of the body force for the pore fluid $\boldsymbol{g} = (g_1, g_2, g_3)^T$, the density ρ_f and the viscosity η of the fluid; the displacement vector $\boldsymbol{u} = (u_1, u_2, u_3)^T$ of the solid phase, the volume density of the body force for the saturated porous medium $\boldsymbol{f} = (f_1, f_2, f_3)^T$, the density ρ_s of the elastic skeleton; the porosity $0 < \phi < 1$, the electrical formation factor F_e, the steady-flow limit of the permeability κ_0, the Lamé coefficients λ, μ of the elastic skeleton and the two Biot coefficients c and m of the saturated matrix; ω_c is the circular frequency at which viscous boundary layers first develop, f_c is the transition frequency, P is the Pride number (typically $P \approx 1/2$), $\boldsymbol{I}$ is the 3×3-identity matrix, and, finally, the superscript ^{T} denotes the transpose; see [15, 19] for details.

Remark 2.1. *The convolution term in the second equations of* (1) *represents the* $1/2$*-derivative of function* $\partial_t^2 \boldsymbol{w} + \omega_c \partial_t \boldsymbol{w}$.

Remark 2.2. *In the low-frequency regime the dissipation efforts in the second equation of* (4) *are given by*

$$\frac{\eta}{\kappa_0} \partial_t \boldsymbol{w},$$

i.e., we obtain the classical Biot equations. Thus, the formulas, which will be constructed in our research, could be used in this case.

3 Statement of the problem

Consider wave propagation in a porous medium $\mathcal{R} = \cup_{k=0}^{N} \mathcal{R}_k$, composed by stratified layers identified with $\mathcal{R}_k = \{\boldsymbol{x} = (x_1, x_2, x_3 \equiv z) \in \mathbb{R}^3 : z_k < z < z_{k+1}\}$, with $0 = z_0 < z_1 \cdots < z_N < z_{N+1} = \infty$. We shall use the coordinate system with positive z-direction downward. The Biot-JKD equations (1)–(3) in the time frequency (ω) domain, at each point $\boldsymbol{x} \in \mathcal{R}$, are (time dependence of $e^{-i\omega t}$ is assumed)

$$\begin{aligned} &- i\omega(\rho \boldsymbol{v} + \rho_f \boldsymbol{q}) = \nabla \cdot \boldsymbol{\tau} + \boldsymbol{f}, \\ &- i\omega(\rho_f \boldsymbol{v} + \rho_w \boldsymbol{q}) + \vartheta \boldsymbol{q} = -\nabla p + \boldsymbol{g}, \\ &- i\omega \boldsymbol{\tau} = (\lambda \nabla \cdot \boldsymbol{v} + c \nabla \cdot \boldsymbol{q}) \boldsymbol{I} + \mu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T), \\ &- i\omega p = -c \nabla \cdot \boldsymbol{v} - m \nabla \cdot \boldsymbol{q}, \end{aligned} \tag{4}$$

where $\boldsymbol{v} = -i\omega \boldsymbol{u}, \boldsymbol{q} = -i\omega \boldsymbol{w}$ are the solid and relative fluid velocities, and

$$\vartheta = \frac{\eta}{\kappa_0} \frac{1}{\sqrt{\omega_c}} (\omega_c - i\omega)^{1/2}. \tag{5}$$

We assume that all material parameters are represented by piecewise constant functions depended only the depth coordinate z, with the discontinuities at the points $z = z_k$, $k = 1, 2, \ldots, N$.

At the internal layer boundaries $z = z_k$, we suppose that the following functions are continuous:

$$\boldsymbol{v},\ \boldsymbol{q},\ p,\ \tau_{13},\ \tau_{23},\ \tau_{33}. \tag{6}$$

The boundary conditions at the free surface $z = 0$ are

$$p = \tau_{13} = \tau_{23} = \tau_{33} = 0. \tag{7}$$

And finally, at the infinity the solution satisfy the following radiation conditions:

$$\lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{v} = \lim_{|\boldsymbol{x}| \to \infty} \boldsymbol{q} = \boldsymbol{0}. \tag{8}$$

4 Method

In this section we discuss a technique that can be used in stratified media, where the model parameters only vary as a function of one coordinate, z. As a result of this simplification, transform methods can be used to reduce the partial differential equations of motion and constitutive laws to a system of ordinary differential equations. This is more easily solved than partial differential equations. The inversion of the results obtained in the frequency domain will give the response in the original domain.

4.1 Special format

Consider the Fourier transform for the variables x_1, x_2

$$\hat{X}(k_1, k_2, z) = F_{x_1 x_2}(X) \equiv \int_{\mathbb{R}^2} e^{-i(k_1 x_1 + k_2 x_2)} X(x_1, x_2, z) dx_1 dx_2.$$

We can apply the Fourier transform $F_{x_1x_2}$ to (4). After substituting the derivatives on x_1, x_2 with the coefficients ik_1, ik_2, the system (4) can be given as a system of ordinary differential equations (ODE's), represented in the terms of $\hat{\boldsymbol{f}}, \hat{\boldsymbol{g}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{q}}, \hat{\boldsymbol{\tau}}, \hat{p}$.

Let $(k_1, k_2)^T$ be the horizontal wave number and $k = \sqrt{k_1^2 + k_2^2}, \gamma = k\omega^{-1}$. For each k, plane wave sources of the form $e^{i(k_1x_1+k_2x_2)}\hat{\boldsymbol{f}}$, $e^{i(k_1x_1+k_2x_2)}\hat{\boldsymbol{g}}$ will produce plane wave responses with spatial dependence of the form $e^{i(k_1x_1+k_2x_2)}$. These equations can be simplified if we rotate to a coordinate system $(\hat{x}_1, \hat{x}_2, \hat{x}_3)^T$ with the first coordinate oriented in the direction of the horizontal wave number, so that all of these plane waves have a spatial dependence of the form $e^{ik\hat{x}_1}$. Therefore, let

$$\boldsymbol{\Omega} = \frac{1}{k}\begin{pmatrix} k_1 & k_2 & 0 \\ -k_2 & k_1 & 0 \\ 0 & 0 & k \end{pmatrix}. \tag{9}$$

The ODE's obtained can be simplified if we define

$$\tilde{\boldsymbol{x}} = \boldsymbol{\Omega x},\ \tilde{\boldsymbol{v}} = \boldsymbol{\Omega}\hat{\boldsymbol{v}},\ \tilde{\boldsymbol{q}} = \boldsymbol{\Omega}\hat{\boldsymbol{q}},\ \tilde{\boldsymbol{\tau}} = \boldsymbol{\Omega}\hat{\boldsymbol{\tau}}\boldsymbol{\Omega}^T,\ \tilde{p} = \hat{p},\ \tilde{\boldsymbol{f}} = \boldsymbol{\Omega}\hat{\boldsymbol{f}},\ \tilde{\boldsymbol{g}} = \boldsymbol{\Omega}\hat{\boldsymbol{g}}. \tag{10}$$

A straightforward calculation uncouples this system

$$\frac{d\boldsymbol{\Phi}^{(m)}}{dz} = -i\omega \boldsymbol{M}^{(m)}\boldsymbol{\Phi}^{(m)} + \boldsymbol{F}^{(m)},\ m = 1, 2, \tag{11}$$

where $\boldsymbol{\Phi}^{(m)}$, $m = 1, 2$, are the $2n_m$-vectors $(n_1 = 3, n_2 = 1)$ defined as

$$\boldsymbol{\Phi}^{(1)} = (\tilde{v}_3, \tilde{\tau}_{13}, -\tilde{q}_3, \tilde{\tau}_{33}, \tilde{v}_1, \tilde{p})^T,\ \boldsymbol{\Phi}^{(2)} = (\tilde{v}_2, \tilde{\tau}_{23})^T, \tag{12}$$

$\boldsymbol{F}^{(m)}$ are the source $2n_m$-vectors, and the $2n_m \times 2n_m$-matrices $\boldsymbol{M}^{(m)}$ have the blocked structure,

$$\boldsymbol{M}^{(m)} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{M}_1^{(m)} \\ \boldsymbol{M}_2^{(m)} & \boldsymbol{0} \end{pmatrix} \tag{13}$$

composed of the $n_m \times n_m$ symmetric sub-matrices $\boldsymbol{M}_1^{(m)}, \boldsymbol{M}_2^{(m)}$. For Systems 1 and 2 the sub-matrices and the corresponding source vectors are

$$\boldsymbol{M}_1^{(1)} = \begin{pmatrix} -\beta m & \beta\gamma(c^2-\lambda m) & -\beta c \\ \beta\gamma(c^2-\lambda m) & \rho + \frac{i\omega\rho_f^2}{\vartheta - i\omega\rho_\omega} - 4\beta\gamma^2\mu(c^2 - m(\lambda+\mu)) & 2\beta\gamma\mu c - \frac{i\omega\rho_f\gamma}{\vartheta - i\omega\rho_\omega} \\ -\beta c & 2\beta\gamma\mu c - \frac{i\omega\rho_f\gamma}{\vartheta - i\omega\rho_\omega} & -\beta(\lambda+2\mu) + \frac{i\omega\gamma^2}{\vartheta - i\omega\rho_\omega} \end{pmatrix}$$

$$\boldsymbol{M}_2^{(1)} = \begin{pmatrix} \rho & \gamma & -\rho_f \\ \gamma & \mu^{-1} & 0 \\ -\rho_f & 0 & -\frac{\vartheta - i\omega\rho_\omega}{i\omega} \end{pmatrix}, \ \boldsymbol{F}^{(1)} = (0, -\tilde{f}_1 - \frac{i\omega\rho_f}{\vartheta - i\omega\rho_\omega}\tilde{g}_1, \frac{ik}{\vartheta - i\omega\rho_\omega}\tilde{g}_1, -\tilde{f}_3, 0, \tilde{g}_3)^T \tag{14}$$

and

$$\boldsymbol{M}_1^{(2)} = \mu^{-1},\ \boldsymbol{M}_2^{(2)} = \rho - \mu\gamma^2 + \frac{i\omega\rho_f^2}{\vartheta - i\omega\rho_\omega},\ \boldsymbol{F}^{(2)} = (0, -\tilde{f}_2 - \frac{i\omega\rho_f}{\vartheta - i\omega\rho_\omega}\tilde{g}_2)^T. \tag{15}$$

Here $\beta = (c^2 - m(\lambda+2\mu))^{-1}$. Once $\boldsymbol{\Phi}^{(1)}$ and $\boldsymbol{\Phi}^{(2)}$ have been determined, we may compute

$$\begin{gathered} \tilde{q}_1 = \frac{1}{\vartheta - i\omega\rho_\omega}(-ik\tilde{p} + i\omega\rho_f\tilde{v}_1 + \tilde{g}_1),\ \tilde{q}_2 = \frac{1}{\vartheta - i\omega\rho_\omega}(i\omega\rho_f\tilde{v}_2 + \tilde{g}_2), \\ \tilde{\tau}_{11} = \beta\big(-4\gamma\mu(c^2 - m(\lambda+\mu))\tilde{v}_1 + (c^2-\lambda m)\tilde{\tau}_{33} + 2\mu c\tilde{p}\big), \\ \tilde{\tau}_{12} = -\mu\gamma\tilde{v}_2,\ \tilde{\tau}_{22} = \beta\big(-2\gamma\mu(c^2-\lambda m)\tilde{v}_1 + (c^2-\lambda m)\tilde{\tau}_{33} + 2\mu c\tilde{p}\big). \end{gathered} \tag{16}$$

The boundary conditions for Systems 1 and 2 at the free surface $z = 0$ are

$$\tilde{p} = \tilde{\tau}_{13} = \tilde{\tau}_{23} = \tilde{\tau}_{33} = 0. \tag{17}$$

Note that (17) gives $n_1 = 3$ conditions for System 1 having $2n_1 = 6$ variables, and gives $n_2 = 1$ condition for System 2, which has $2n_2 = 2$ variables. It means that for each system we need n_m, $m = 1, 2$ additional conditions to completely specify the solution. These relations we obtain using the radiation condition (8), which means that there are no up-going waves from $z = \infty$.

4.2 Diagonalization

In order to decompose the solution of system (11) into up- and down-going waves we need to diagonalize the matrix $\boldsymbol{M}^{(m)}$ given by (13). The diagonalization procedure is based on the Ursin formalism [20].

For simplicity we drop the superscript $^{(m)}$. Assume that $\boldsymbol{M}_1\boldsymbol{M}_2$ has n distinct nonzero eigenvalues λ_j^2, $j = 1, 2, \dots, n$, with associated normalized eigenvectors $\boldsymbol{a}_j$, i.e.,

$$\boldsymbol{M}_1\boldsymbol{M}_2\boldsymbol{a}_j = \lambda_j^2\boldsymbol{a}_j :\ \boldsymbol{a}_j^T\boldsymbol{M}_2\boldsymbol{a}_j = \lambda_j,\ j = 1, 2, \dots, n. \tag{18}$$

Here $\lambda_j = \sqrt{\lambda_j^2}$ with the branch chosen so that $\mathrm{Im}(\lambda_j) \geq 0$ and $\lambda_j > 0$ if λ_j is real. This branch of the square root will be adapted through the paper. Define

$$\boldsymbol{b}_j = \lambda_j^{-1}\boldsymbol{M}_2\boldsymbol{a}_j. \tag{19}$$

Multiplying (19) by $\boldsymbol{M}_2\boldsymbol{M}_1$ and using (18) we can conclude that this vector is an eigenvector of $\boldsymbol{M}_2\boldsymbol{M}_1$ with eigenvalue λ_j^2. Using symmetry of matrices $\boldsymbol{M}_1, \boldsymbol{M}_2$ it is easy to show that $\boldsymbol{b}_j$ is a left eigenvector of $\boldsymbol{M}_1\boldsymbol{M}_2$ and

$$\lambda_i^2 \boldsymbol{a}_j^T \boldsymbol{b}_i = \boldsymbol{a}_j^T \boldsymbol{M}_2 \boldsymbol{M}_1 \boldsymbol{b}_i = b_i^T \boldsymbol{M}_1 \boldsymbol{M}_2 \boldsymbol{a}_j = \lambda_j^2 \boldsymbol{a}_j^T \boldsymbol{b}_i. \tag{20}$$

The latter implies $\boldsymbol{a}_j^T \boldsymbol{b}_i = 0$ if $j \neq i$. Thus, using (18) we arrive at

$$\boldsymbol{a}_j^T \boldsymbol{b}_i = \delta_{j,i}, \tag{21}$$

where $\delta_{j,i}$ is the Kronecker delta.

Now consider the $n \times n$-matrices $\boldsymbol{L}_1$ (whose j-th column is $\boldsymbol{a}_j$) and $\boldsymbol{L}_2$ (whose i-th column is $\boldsymbol{b}_i$). Using (21) we can prove that matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ posses the symmetries

$$\boldsymbol{L}_2^T \boldsymbol{L}_1 = \boldsymbol{L}_2 \boldsymbol{L}_1^T = \boldsymbol{I}. \tag{22}$$

Relations (18), (19) and (22) imply

$$\boldsymbol{M}_1 = \boldsymbol{L}_1 \boldsymbol{\Lambda} \boldsymbol{L}_1^T, \; \boldsymbol{M}_2 = \boldsymbol{L}_2 \boldsymbol{\Lambda} \boldsymbol{L}_2^T, \tag{23}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Let $\tilde{\boldsymbol{\Lambda}} = \mathrm{diag}(\boldsymbol{\Lambda}, -\boldsymbol{\Lambda})$. Using (22) and (23) we obtain

$$\boldsymbol{M} = \boldsymbol{L} \tilde{\boldsymbol{\Lambda}} \boldsymbol{L}^{-1}, \tag{24}$$

where

$$\boldsymbol{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{L}_1 & \boldsymbol{L}_1 \\ \boldsymbol{L}_2 & -\boldsymbol{L}_2 \end{pmatrix}, \; \boldsymbol{L}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{L}_2^T & \boldsymbol{L}_1^T \\ \boldsymbol{L}_2^T & -\boldsymbol{L}_1^T \end{pmatrix}.$$

The explicit algebraic formulas for $\lambda_j, \boldsymbol{a}_j, \boldsymbol{b}_j$, are:

System 1. There are three modes: fast compressional wave ($\lambda_1^{(1)}$), Biot slow wave ($\lambda_2^{(1)}$) and vertical shear wave ($\lambda_3^{(1)}$).

$$(\lambda_j^{(1)})^2 = -\gamma^2 + \beta\big(c\rho_f - \frac{m\rho}{2} + \frac{(\lambda+2\mu)(\vartheta - i\omega\rho_\omega)}{2i\omega}\big) \pm$$
$$\pm \frac{\beta}{2}\sqrt{\big(m\rho + \frac{(\lambda+2\mu)(\vartheta - i\omega\rho_\omega)}{i\omega}\big)^2 - 4(m\rho_f + c\frac{\vartheta - i\omega\rho_\omega}{i\omega})(c\rho - (\lambda+2\mu)\rho_f)},$$

$j = 1, 2,$ with (+) for $j = 1$ and (-) for $j = 2$, and $(\lambda_3^{(1)})^2 = -\gamma^2 + \mu^{-1}\big(\rho + \dfrac{i\omega\rho_f^2}{\vartheta - i\omega\rho_\omega}\big)$;

$$\boldsymbol{a}_j^{(1)} = \overline{a}_j(-1, 2\mu\gamma, \xi_j)^T, \; j = 1, 2, \; \boldsymbol{a}_3^{(1)} = \frac{\overline{a}_3}{\lambda_3^{(1)}}\big(\gamma, \mu(\lambda_3^{(1)})^2 - \mu\gamma^2, -\frac{i\omega\gamma\rho_f}{\vartheta - i\omega\rho_\omega}\big)^T,$$

$$\boldsymbol{b}_j^{(1)} = \frac{\overline{a}_j}{\lambda_j^{(1)}}\big(2\mu\gamma^2 - \rho - \rho_f\xi_j, \gamma, \rho_f - \xi_j\frac{\vartheta - i\omega\rho_\omega}{i\omega}\big)^T, \; j = 1, 2, \; \boldsymbol{b}_3^{(1)} = \overline{a}_3(2\mu\gamma, 1, 0)^T,$$

where

$$\xi_j = \frac{c\rho - (\lambda+2\mu)\rho_f}{\frac{(\lambda_j^{(1)})^2 + \gamma^2}{\beta} - c\rho_f + i(\lambda+2\mu)\frac{\vartheta - i\omega\rho_\omega}{i\omega}}, \; j = 1, 2,$$

$$\overline{a}_j = \sqrt{\frac{\lambda_j^{(1)}}{\rho + 2\rho_f\xi_j + i\xi_j^2\frac{\vartheta - i\omega\rho_\omega}{i\omega}}}, \; j = 1, 2, \; \overline{a}_3 = \sqrt{\frac{\lambda_3^{(1)}}{\mu(\lambda_3^{(1)})^2 + \mu\gamma^2}}. \tag{25}$$

System 2. There is the horizontal shear wave ($\lambda^{(2)}$) only.

$$(\lambda^{(2)})^2 = -\gamma^2 + \mu^{-1}\big(\rho + \frac{i\omega\rho_f^2}{\vartheta - i\omega\rho_\omega}\big),\ \boldsymbol{a}^{(2)} = \sqrt{\frac{1}{\mu\lambda^{(2)}}},\ \boldsymbol{b}^{(2)} = \sqrt{\mu\lambda^{(2)}}. \tag{26}$$

Using formulas (25) and (26) matrices $\boldsymbol{L}_1, \boldsymbol{L}_2, \boldsymbol{L}, \boldsymbol{L}^{-1}$ and $\boldsymbol{\Lambda}$ can be calculated rapidly.

4.3 Reflection and transmission coefficient matrices

First consider a homogeneous source-free region of space. Dropping $^{(m)}$ we have a $2n$-dimensional system of the form (11) with $\boldsymbol{M}$ constant and $\boldsymbol{F} = \boldsymbol{0}$. In order to decompose vector $\boldsymbol{\Phi}$ into up- and down-going waves we make the linear transformation

$$\boldsymbol{\Phi} = \boldsymbol{L}\boldsymbol{\Psi} \tag{27}$$

with $\boldsymbol{\Psi} = (\boldsymbol{U}, \boldsymbol{D})^T$, where $\boldsymbol{U}, \boldsymbol{D}$ are n-vectors. Using (11) (with $\boldsymbol{F} = \boldsymbol{0}$) and (27), we arrive at

$$\frac{d}{dz}\boldsymbol{\Psi} = -i\omega\tilde{\boldsymbol{\Lambda}}\boldsymbol{\Psi}.$$

Then

$$\boldsymbol{\Psi}(z) = \big(e^{-i\omega\boldsymbol{\Lambda}(z-z_0)}\boldsymbol{U}(z_0), e^{i\omega\boldsymbol{\Lambda}(z-z_0)}\boldsymbol{D}(z_0)\big)^T, \tag{28}$$

where z_0 is a fixed point in the same source-free region of space as z. The quantities $e^{\pm i\omega\boldsymbol{\Lambda}(z-z_0)}$ are diagonal matrices with j-th diagonal element equal to $e^{\pm i\omega\lambda_j(z-z_0)}$, respectively. Therefore, the vectors $\boldsymbol{U}, \boldsymbol{D}$ represent up-going ($\boldsymbol{U}$) and down-going ($\boldsymbol{D}$) waves.

Next consider an interface at $z = \overline{z}$, where the material parameters vary discontinuously across $\overline{z}$. We denote by $^\pm$ quantities evaluated at $\overline{z}^\pm = \overline{z} \pm 0$. Since $\boldsymbol{\Phi}$ is continuous across $\overline{z}$, we obtain from (27) that

$$\boldsymbol{\Psi}^+ = \boldsymbol{J}\boldsymbol{\Psi}^-,\ \boldsymbol{\Psi}^- = \boldsymbol{J}^{-1}\boldsymbol{\Psi}^+ \tag{29}$$

with the jump matrix

$$\boldsymbol{J} = (\boldsymbol{L}^+)^{-1}\boldsymbol{L}^- = \begin{pmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_2 \\ \boldsymbol{J}_2 & \boldsymbol{J}_1 \end{pmatrix},\ \boldsymbol{J}^{-1} = \begin{pmatrix} \boldsymbol{J}_1^T & -\boldsymbol{J}_2^T \\ -\boldsymbol{J}_2^T & \boldsymbol{J}_1^T \end{pmatrix},$$

where $\boldsymbol{J}_1, \boldsymbol{J}_2$ are the $n \times n$-matrices defined as follows

$$\boldsymbol{J}_1 = \frac{1}{2}\big[\big(\boldsymbol{L}_2^+\big)^T\boldsymbol{L}_1^- + \big(\boldsymbol{L}_1^+\big)^T\boldsymbol{L}_2^-\big],\ \boldsymbol{J}_2 = \frac{1}{2}\big[\big(\boldsymbol{L}_2^+\big)^T\boldsymbol{L}_1^- - \big(\boldsymbol{L}_1^+\big)^T\boldsymbol{L}_2^-\big].$$

Consider a stack of homogeneous layers $0 = z_0 < z_1 < \cdots < z_N < z_{N+1} = \infty$. Homogeneous, source-free region is assumed in the layer number j, i.e., for $z_j < z < z_{j+1}$. We denote by subscript j a quantity at interface $z = z_j$, with superscripts $^\pm$ as before. Using (27) and (29) we obtain

$$\big(\boldsymbol{U}_N^-, \boldsymbol{D}_N^-\big)^T = \boldsymbol{J}_N^{-1}\big(\boldsymbol{0}, \boldsymbol{D}_N^+\big)^T,$$

where we have used that there is no up-going wave below the last interface at $z = z_N$. Thus

$$\boldsymbol{U}_N^- = \boldsymbol{\Gamma}_N \boldsymbol{D}_N^-, \ \boldsymbol{D}_N^+ = \boldsymbol{T}_N \boldsymbol{D}_N^-, \tag{30}$$

where

$$\boldsymbol{\Gamma}_N = -\boldsymbol{J}_{2,N}^T \big(\boldsymbol{J}_{1,N}^T\big)^{-1}, \ \boldsymbol{T}_N = \big(\boldsymbol{J}_{1,N}^T\big)^{-1}. \tag{31}$$

Here $\boldsymbol{\Gamma}_N$ is the reflection matrix and $\boldsymbol{T}_N$ is the transmission matrix from the last interface $z = z_N$. These matrices can be used to compute the reflected (up-going) and transmitted (down-going) waves from the top of this interface and beneath this interface, respectively, when the incident wave is known.

Let $j < N$ and $\triangle z_j = z_{j+1} - z_j$, $j = 1, 2, \ldots, N-1$, is the thickness of layer number j. Then by jumping across the layer boundary and using (28), (29) we obtain

$$\begin{aligned} \boldsymbol{U}_j^- &= \boldsymbol{J}_{1,j}^T e^{i\omega \boldsymbol{\Lambda}_j \triangle z_j} \boldsymbol{U}_{j+1}^- - \boldsymbol{J}_{2,j}^T e^{-i\omega \boldsymbol{\Lambda}_j \triangle z_j} \boldsymbol{D}_{j+1}^-, \\ \boldsymbol{D}_j^- &= -\boldsymbol{J}_{2,j}^T e^{i\omega \boldsymbol{\Lambda}_j \triangle z_j} \boldsymbol{U}_{j+1}^- + \boldsymbol{J}_{1,j}^T e^{-i\omega \boldsymbol{\Lambda}_j \triangle z_j} \boldsymbol{D}_{j+1}^-. \end{aligned} \tag{32}$$

Define reflection and transmission matrices $\boldsymbol{\Gamma}_j, \boldsymbol{T}_j$ by the relations that for any incident wave $\boldsymbol{D}_j^-$ at the top of stack of layers underlying $z = z_j$

$$\boldsymbol{U}_j^- = \boldsymbol{\Gamma}_j \boldsymbol{D}_j^-, \ \boldsymbol{D}_j^+ = \boldsymbol{T}_j \boldsymbol{D}_j^-. \tag{33}$$

Therefore $\boldsymbol{\Gamma}_j$ computes the reflected wave from the stack and $\boldsymbol{T}_j$ computes the transmitted wave below the stack, when the incident wave is known. From (32), (33) we obtain by induction

$$\begin{aligned} \boldsymbol{\Gamma}_j &= \big(\boldsymbol{J}_{1,j}^T \tilde{\boldsymbol{\Gamma}}_{j+1} - \boldsymbol{J}_{2,j}^T\big)\big(- \boldsymbol{J}_{2,j}^T \tilde{\boldsymbol{\Gamma}}_{j+1} + \boldsymbol{J}_{1,j}^T\big)^{-1}, \\ \boldsymbol{T}_j &= \boldsymbol{T}_{j+1} e^{i\omega \boldsymbol{\Lambda}_j \triangle z_j} \big(- \boldsymbol{J}_{2,j}^T \tilde{\boldsymbol{\Gamma}}_{j+1} + \boldsymbol{J}_{1,j}^T\big)^{-1}, \end{aligned} \tag{34}$$

where $\tilde{\boldsymbol{\Gamma}}_{j+1} = e^{i\omega \boldsymbol{\Lambda}_j \triangle z_j} \boldsymbol{\Gamma}_{j+1} e^{i\omega \boldsymbol{\Lambda}_j \triangle z_j}$. Finally, by induction we can prove that $\boldsymbol{\Gamma}_j$ is symmetric.

Thus all the reflection and transmission matrices can be calculated by (34), starting with (31).

4.4 Sources and boundary conditions

Consider a $2n$-dimensional system of the form (11) with $^{(m)}$ omitted. Let the source be of the form

$$\boldsymbol{F} = \boldsymbol{F}_0 \delta(z - z_s) + \boldsymbol{F}_1 \delta'(z - z_s) \tag{35}$$

with $\boldsymbol{F}_0, \boldsymbol{F}_1$ independent of z. Here δ is the Dirac function and z_s is a source position. Let

$$\boldsymbol{\Phi}_0 = \boldsymbol{\Phi} - \boldsymbol{F}_1 \delta(z - z_s). \tag{36}$$

Using (11), (35) and (36), we obtain

$$\frac{d\boldsymbol{\Phi}_0}{dz} = -i\omega \boldsymbol{M} \boldsymbol{\Phi}_0 + \big(\boldsymbol{F}_0 - i\omega \boldsymbol{M} \boldsymbol{F}_1\big)\delta(z - z_s). \tag{37}$$

Define n-vectors $\boldsymbol{S}_1, \boldsymbol{S}_2$ by the following formula

$$\big(\boldsymbol{S}_1, \boldsymbol{S}_2\big)^T = i\omega \boldsymbol{M}\boldsymbol{F}_1 - \boldsymbol{F}_0. \tag{38}$$

Integrating (37) over interval (z_s^-, z_s^+) and using (36) and (38), we arrive at

$$\boldsymbol{\Phi}(z_s^-) = \boldsymbol{\Phi}(z_s^+) + \big(\boldsymbol{S}_1, \boldsymbol{S}_2\big)^T. \tag{39}$$

Now, inserting a fictitious layer boundary at $z = z_s^+$ and using the previously obtained results, we are able to compute the reflection matrix $\boldsymbol{\Gamma}_s = \boldsymbol{\Gamma}(z_s^+)$ from the top of this layer. Since the material properties do not change at z_s, the equalities $\boldsymbol{J}_1 = \boldsymbol{I}, \boldsymbol{J}_2 = \boldsymbol{0}$ are valid at z_s^+. Then the up-going wave $\boldsymbol{U}_s = \boldsymbol{U}_s(z_s^+)$ is related to the down-going wave $\boldsymbol{D}_s = \boldsymbol{D}_s(z_s^+)$ there by (33). Thus

$$\boldsymbol{\Psi}(z_s^+) = \big(\boldsymbol{\Gamma}_s \boldsymbol{D}_s, \boldsymbol{D}_s\big)^T. \tag{40}$$

Using (27), (39) and (40) we finally arrive at

$$\boldsymbol{\Psi}(z_s^-) = \big(\boldsymbol{\Gamma}_s \boldsymbol{D}_s, \boldsymbol{D}_s\big)^T + \frac{1}{\sqrt{2}}\big(\boldsymbol{L}_2^T \boldsymbol{S}_1 + \boldsymbol{L}_1^T \boldsymbol{S}_2, \boldsymbol{L}_2^T \boldsymbol{S}_1 - \boldsymbol{L}_1^T \boldsymbol{S}_2\big)^T. \tag{41}$$

This expression may now propagated upwards through layers, using (28) and jumped upwards across layers boundaries using (29) until we reach the free surface at $z = 0^+$. Then the n boundary conditions at $z = 0$ can be used to find the n unknowns $\boldsymbol{D}_s$.

Consider now one particular case when $z_s \in (0, z_1)$. In this case

$$\begin{aligned}\boldsymbol{\Psi}(0^+) &= \big(e^{i\omega\boldsymbol{\Lambda}z_s}\boldsymbol{\Gamma}_s \boldsymbol{D}_s, e^{-i\omega\boldsymbol{\Lambda}z_s}\boldsymbol{D}_s\big)^T \\ &+ \frac{1}{\sqrt{2}}\big(e^{i\omega\boldsymbol{\Lambda}z_s}(\boldsymbol{L}_2^T \boldsymbol{S}_1 + \boldsymbol{L}_1^T \boldsymbol{S}_2), e^{-i\omega\boldsymbol{\Lambda}z_s}(\boldsymbol{L}_2^T \boldsymbol{S}_1 - \boldsymbol{L}_1^T \boldsymbol{S}_2)\big)^T.\end{aligned} \tag{42}$$

We next write

$$\boldsymbol{\Phi}(0^+) = \big(\boldsymbol{G}_1\boldsymbol{\Phi}_0, \boldsymbol{G}_2\boldsymbol{\Phi}_0\big)^T, \tag{43}$$

where $\boldsymbol{\Phi}_0$ is now an n-vector of unknowns at $z = 0$ and $\boldsymbol{G}_1, \boldsymbol{G}_2$ are $n \times n$ matrices. For System 1, let

$$\boldsymbol{\Phi}_0^{(1)} = \big(\tilde{v}_3, -\tilde{q}_3, \tilde{v}_1\big)^T_{z=0^+},\ \boldsymbol{G}_1^{(1)} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\ \boldsymbol{G}_2^{(1)} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}. \tag{44}$$

We can check that (43) holds for System 1 with the boundary conditions given by (17). For System 2, let

$$\boldsymbol{\Phi}_0^{(2)} = \tilde{v}_2(0^+),\ \boldsymbol{G}_1^{(2)} = 1,\ \boldsymbol{G}_2^{(2)} = 0. \tag{45}$$

It may be checked that (43) holds for System 2 with the boundary conditions given by (17). Using (27), (42) and (43) we obtain

$$\begin{aligned}\boldsymbol{\Phi}_0 &= \Big(e^{i\omega\boldsymbol{\Lambda}z_s}\boldsymbol{\Gamma}_s e^{i\omega\boldsymbol{\Lambda}z_s}\big(\boldsymbol{L}_2^T\boldsymbol{G}_1 - \boldsymbol{L}_1^T\boldsymbol{G}_2\big) - \big(\boldsymbol{L}_2^T\boldsymbol{G}_1 + \boldsymbol{L}_1^T\boldsymbol{G}_2\big)\Big)^{-1} \\ &\times e^{i\omega\boldsymbol{\Lambda}z_s}\Big(\boldsymbol{\Gamma}_s\big(\boldsymbol{L}_2^T\boldsymbol{S}_1 - \boldsymbol{L}_1^T\boldsymbol{S}_2\big) - \big(\boldsymbol{L}_2^T\boldsymbol{S}_1 + \boldsymbol{L}_1^T\boldsymbol{S}_2\big)\Big), \\ \boldsymbol{D}_s &= \frac{1}{\sqrt{2}}e^{i\omega\boldsymbol{\Lambda}z_s}\big(\boldsymbol{L}_2^T\boldsymbol{G}_1 - \boldsymbol{L}_1^T\boldsymbol{G}_2\big)\boldsymbol{\Phi}_0 - \frac{1}{\sqrt{2}}\big(\boldsymbol{L}_2^T\boldsymbol{S}_1 - \boldsymbol{L}_1^T\boldsymbol{S}_2\big).\end{aligned} \tag{46}$$

In particular, when the source is situated just below the surface we get

$$\begin{aligned}\boldsymbol{\Phi}_0 = &\Big(\big(\boldsymbol{\Gamma}_s - \boldsymbol{I}\big)\boldsymbol{L}_2^T\boldsymbol{G}_1 - \big(\boldsymbol{\Gamma}_s + \boldsymbol{I}\big)\boldsymbol{L}_1^T\boldsymbol{G}_2\Big)^{-1} \\ &\times \Big(\big(\boldsymbol{\Gamma}_s - \boldsymbol{I}\big)\boldsymbol{L}_2^T\boldsymbol{S}_1 - \big(\boldsymbol{\Gamma}_s + \boldsymbol{I}\big)\boldsymbol{L}_1^T\boldsymbol{S}_2\Big) \text{ as } z_s \to 0^+.\end{aligned} \tag{47}$$

$\boldsymbol{\Phi}_0$ defines all of $\boldsymbol{\Phi}$ at the free surface, and $\boldsymbol{D}_s, \boldsymbol{U}_s = \boldsymbol{\Gamma}_s\boldsymbol{D}_s$ give all of $\boldsymbol{\Phi}$ just below the source. Now we are able to compute $\boldsymbol{\Phi}$ in any $z \in \mathbb{R}_+$ by propagating through the layers using (28) and (29).

Remark 4.1. *Propagation of an upward-going wave in the downward direction will be unstable numerically using* (28)*, because the complex exponentials grow rather than decay with distance. Therefore, numerically one has to obtain* $\boldsymbol{U}$ *from* $\boldsymbol{D}$ *using the reflection or transmission matrices.*

4.5 Inverse transforms

Inverting (10), we can calculate the hat ($\hat{}$) variables, i.e.,

$$\hat{\boldsymbol{v}} = \boldsymbol{\Omega}^T\tilde{\boldsymbol{v}},\ \hat{\boldsymbol{q}} = \boldsymbol{\Omega}^T\tilde{\boldsymbol{q}},\ \hat{\boldsymbol{\tau}} = \boldsymbol{\Omega}^T\tilde{\boldsymbol{\tau}}\boldsymbol{\Omega},\ \hat{p} = \tilde{p}. \tag{48}$$

The matrices for Systems 1 and 2 depend only on the magnitude k. However, factors k_1 and k_2 are introduced by (9) and possibly by the directionality of the source. For any function $\hat{h}(k)$ let

$$\mathcal{T}_{j_1,j_2}\big(\hat{h}\big) \equiv F^{-1}_{x_1x_2}\big(k_1^{j_1}k_2^{j_2}\hat{h}(k)\big) = (-i)^{j_1+j_2}\partial^{j_1}_{x_1}\partial^{j_2}_{x_2}F^{-1}_{x_1x_2}\big(\hat{h}(k)\big).$$

We can now compute these quantities as Hankel transforms in the cylindrical coordinates r, θ, z.

Define

$$B_{j_1,j_2}\big(\hat{h}\big) = \frac{1}{2\pi}\int_0^\infty k^{j_1}J_{j_2}(kr)\hat{h}(k)dk,$$

where J_{j_2} is the Bessel function and j_1, j_2 are nonnegative integers. Then

$$\begin{aligned}&\mathcal{T}_{0,0} = B_{1,0},\ \mathcal{T}_{1,0} = i\cos\theta B_{2,1},\ \mathcal{T}_{0,1} = i\sin\theta B_{2,1}, \\ &\mathcal{T}_{1,1} = \sin\theta\cos\theta\big(B_{3,0} - \frac{2}{r}B_{2,1}\big),\ \mathcal{T}_{2,0} = \cos^2\theta B_{3,0} - \frac{\cos 2\theta}{r}B_{2,1}, \\ &\mathcal{T}_{0,2} = \sin^2\theta B_{3,0} + \frac{\cos 2\theta}{r}B_{2,1}.\end{aligned} \tag{49}$$

5 Source implementation examples

The basic elastic (seismic) sources are a directional force, a pressure source, and a shear source, simulating, for instance, a vertical vibrator, an explosion, or a shear vibrator. Complex sources can be represented by a set of directional forces. In this Section we consider two typical examples of the source function that generally appears as an additive term in (11).

An explosion source imposed on the solid and the fluid can be defined in the following form

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) = -s(\omega)\nabla\delta(\boldsymbol{x} - \boldsymbol{x}_s),$$

where $\boldsymbol{x}_s = (0, 0, z_s)^T$ is the source position and $s(\omega)$ is the spectrum of the elastic moment. Applying the Fourier transform $F_{x_1x_2}$ we obtain

$$\hat{\boldsymbol{f}} = \hat{\boldsymbol{g}} = -s(\omega)\big(ik_1\delta(z - z_s), ik_2\delta(z - z_s), \delta'(z - z_s)\big)^T.$$

The rotation by $\boldsymbol{\Omega}$ yields

$$\tilde{\boldsymbol{f}} = \tilde{\boldsymbol{g}} = -s(\omega)\big(ik\delta(z - z_s), 0, \delta'(z - z_s)\big)^T. \tag{50}$$

Substitution of (50) into (14) gives the source for System 1 in the form (35) with

$$\begin{aligned} \boldsymbol{F}_0^{(1)} &= s(\omega)\Big(0, ik\big(1 + \frac{i\omega\rho_f}{\vartheta - i\omega\rho_\omega}\big), \frac{k^2}{\vartheta - i\omega\rho_\omega}, 0, 0, 0\Big)^T, \\ \boldsymbol{F}_1^{(1)} &= s(\omega)(0, 0, 0, 1, 0, -1)^T. \end{aligned} \tag{51}$$

Substitution of (50) into (15) shows that $\boldsymbol{F}^{(2)}$ is zero, then $\tilde{v}_2, \tilde{\tau}_{23}$ associated with System 2 are zero too. This is to be expected result because System 2 is related to the horizontal shear waves, which are not excited by the dynamic source. Substitution of (51) into (38) gives

$$\begin{aligned} \boldsymbol{S}_1^{(1)} &= i\beta s(w)\big(\omega(c - m), 2k\mu(m - c), \omega(\lambda + 2\mu - c)\big)^T, \\ \boldsymbol{S}_2^{(1)} &= (0, 0, 0)^T. \end{aligned} \tag{52}$$

Formulas (52) may be used in (46) or (47) for a shallow explosion source, to obtain all the tilde (˜) functions.

To invert the rotation $\boldsymbol{\Omega}$, using (48), note that from (12) and (16) and the vanishing of System 2, $\tilde{v}_2, \tilde{q}_2, \tilde{\tau}_{12}, \tilde{\tau}_{23}$ are identically zero. All the remaining tilde functions depend of k only and can be calculated by the following formulas

$$\begin{aligned} &\hat{v}_1 = \frac{k_1}{k}\tilde{v}_1,\ \hat{v}_2 = \frac{k_2}{k}\tilde{v}_1,\ \hat{v}_3 = \tilde{v}_3,\ \hat{q}_1 = \frac{k_1}{k}\tilde{q}_1,\ \hat{q}_2 = \frac{k_2}{k}\tilde{q}_1,\ \hat{q}_3 = \tilde{q}_3, \\ &\hat{\tau}_{11} = \frac{k_1^2\tilde{\tau}_{11} + k_2^2\tilde{\tau}_{22}}{k^2},\ \hat{\tau}_{12} = \frac{k_1k_2(\tilde{\tau}_{11} - \tilde{\tau}_{22})}{k^2},\ \hat{\tau}_{22} = \frac{k_2^2\tilde{\tau}_{11} + k_1^2\tilde{\tau}_{22}}{k^2}, \\ &\hat{\tau}_{13} = \frac{k_1\tilde{\tau}_{13}}{k},\ \hat{\tau}_{23} = \frac{k_2\tilde{\tau}_{13}}{k},\ \hat{\tau}_{33} = \tilde{\tau}_{33}. \end{aligned} \tag{53}$$

Then the Fourier transform $F_{x_1x_2}$ can be inverted in cylindrical coordinates (r, θ, z) using (49) to obtain the solid and fluid velocities

$$\boldsymbol{v} = iB_{1,1}(\tilde{v}_1)\boldsymbol{e}_r + B_{1,0}(\tilde{v}_3)\boldsymbol{e}_z,\ \boldsymbol{q} = iB_{1,1}(\tilde{q}_1)\boldsymbol{e}_r + B_{1,0}(\tilde{q}_3)\boldsymbol{e}_z \tag{54}$$

and the stress tensor components

$$\begin{aligned}
&\tau_{11} = \mathcal{T}_{2,0}\big(k^{-2}\tilde{\tau}_{11}\big) + \mathcal{T}_{0,2}\big(k^{-2}\tilde{\tau}_{22}\big),\ \tau_{12} = \mathcal{T}_{1,1}\big(k^{-2}(\tilde{\tau}_{11} - \tilde{\tau}_{22})\big),\\
&\tau_{22} = \mathcal{T}_{0,2}\big(k^{-2}\tilde{\tau}_{11}\big) + \mathcal{T}_{2,0}\big(k^{-2}\tilde{\tau}_{22}\big),\ \tau_{13} = \mathcal{T}_{1,0}\big(k^{-1}\tilde{\tau}_{13}\big),\\
&\tau_{23} = \mathcal{T}_{0,1}\big(k^{-1}\tilde{\tau}_{13}\big),\ \tau_{33} = \mathcal{T}_{0,0}(\tilde{\tau}_{33}).
\end{aligned} \tag{55}$$

These stresses may now be computed in cylindrical coordinates from (49) using the Hankel transforms of the appropriate tilde functions.

Next, consider a vertical point force acting on the free surface $z = 0$, i.e.,

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) = (0,0,1)^T s(\omega)\delta(x_1)\delta(x_2)\delta(z - z_s) \text{ as } z_s \to 0^+.$$

This models hammer, weight drop, and vibroseis sources. Applying the Fourier transform $F_{x_1x_2}$ and rotation Ω we arrive at

$$\tilde{\boldsymbol{f}} = \tilde{\boldsymbol{g}} = \hat{\boldsymbol{f}} = \hat{\boldsymbol{g}} = (0,0,1)^T s(\omega)\delta(z - z_s). \tag{56}$$

Substitution of (56) into (14), (15) yields the source for Systems 1 and 2 in the form

$$\boldsymbol{F}^{(1)} = (0,0,0,-1,0,1)^T s(\omega)\delta(z - z_s),\ \boldsymbol{F}^{(2)} = (0,0)^T. \tag{57}$$

Thus, all the variables in System 2 are zero, as it was in the case of the dynamite source. From (35), (38) and (57) we obtain

$$\boldsymbol{S}_1^{(1)} = (0,0,0)^T,\ \boldsymbol{S}_2^{(1)} = (1,0,-1)^T s(\omega). \tag{58}$$

Now all the tilde variables at the free surface may be computed from equations (47) as $z_s \to 0^+$ and propagated anywhere else in space. Note that $\boldsymbol{S}_1^{(1)}, \boldsymbol{S}_2^{(1)}$ are independent of k_1, k_2, so that the tilde variables depend only on k and not on wavenumber direction. Therefore, similar to dynamite we can transform to the hat variables using (53) and transform back to the spatial coordinates using (54) and (55).

4 Conclusion

We have shown how the complete Biot-JKD equations can be put into the Ursin form in a plane-layered medium. We have derived explicit formulas for calculating reflected/transmitted poroelastic waves in the space-frequency domain. The formulas obtained can be used to create a numerical algorithm for the analysis of poroelastic waves across all frequencies.

Acknowledgement

The first author thanks the Program for the Formation of Human Resources in Geophysics - PRH-PB 226 (UENF/Petrobras) for the financial support and the Laboratory of Engineering and Exploration of Petroleum (LENEP/CCT/UENF) for having provided the conditions for this work. The second author is a member of the National Institute of Science and Technology of Petroleum Geophysics (INCT-GP/CNPq/MEC), Brazil.

References

[1] de Boer, R., *Theory of porous media*, Springer-Verlag, Berlin, 2000.

[2] de Boer, R., *Highligts in the historical development of the porous media theory: toward a consistent macroscopic theory*, Appl. Mech. Rev., Vol. 49(4) (1996), 201–262.

[3] Biot, M.A., *Theory of propagation of elastic waves in a fluid-saturated porous media. I. low-frequency range*, J. Acoust. Soc. Am., Vol. 28 (1956), 168–178.

[4] Biot, M.A., *Theory of propagation of elastic waves in a fluid-saturated porous media. II. higher-frequency range*, J. Acoust. Soc. Am., Vol. 28 (1956), 179–191.

[5] Frenkel, J., *On the theory of seismic and seismoelectric phenomena in a moist soil*, J. Phys. (USSR), Vol. 3 (1944), 230–240.

[6] Nikolaevskiy, V., *Biot-Frenkel poromechanics in Russia (review)*, J. Eng. Mech., Vol.131 (2005), 888–897.

[7] Pride, S. and Garambois, S., *Electroseismic wave theory of Frenkel and more recent developments*, J. Eng. Mech., Vol. 131 (2005), 898–907.

[8] Plona, T.J., *Observation of a second bulk compressional wave in a porous medium at ultrasonic frequencies*, Applied Physics Letters, Vol. 36(4) (1980), 259–261.

[9] Schanz, M., *Poroelastodynamics: linear models, analytical solutions, and numerical methods*, Appl. Mech. Rev., Vol. 62 (2009), 030803–030815.

[10] Auriault, J., Borne, L., and Chambon R., *Dynamics of porous saturated media*, J. Acoust. Soc. Am., Vol. 77 (1985), 1641–1650.

[11] Johnson, D.L., Koplik, J., and Dashen, R., *Theory of dynamic permeability and tortuosity in fluid-saturated porous media*, Journal of Fluid Mechanics, Vol. 176(1) (1987), 379–402.

[12] Carcione, J.M., *Wave propagation in anisotropic, saturated porous media: plane-wave theory and numerical simulation*, J. Acoust. Soc. Am., Vol. 99 (1996), 2655–2666.

[13] Hanyga, A. and Lu, J.-F., *Wave field simulation for heterogeneous transversely isotropic porous media with the JKD dynamic permeability*, Computational Mechanics, Vol. 36(3) (2005), 196–208.

[14] Masson, Y.J. and Pride, S.R., *Finite-difference modeling of Biot's poroelastic equations across all frequencies*, Geophysics, Vol. 75(5) (2010), N33–N41.

[15] Blanc, E., *Time-domain numerical modeling of poroelastic waves: the Biot-JKD model with fractional derivatives*, PhD thesis, Aix-Marseille University, France, 2014.

[16] Li, Z., Kouri, D.J., and Chesnokov, E.M., *Time-domain solution of poroelastic wave equation with dynamic permeability*, SEG New Orleans Annual Meeting 2015, 3589–3593.

[17] Milani, M., Monachesi, L., Sabbione, J.I. , Rubino, J.G., and Holliger, K., *A generalized effective anisotropic poroelastic model for periodically layered media accounting for both Biot's global and interlayer flows*, Geophysical Prospecting, Vol. 64(4) (2016), 1135–1148.

[18] Carcione, J.M., Morency, C., and Santos, J.E., *Computational poroelasticity - a review*, Geophysics, Vol. 75(5) (2010), 75A229–75A243.

[19] Lorenzi, A. and Priimenko, V., *Direct problems for poroelastic waves with fractional derivatives*, SIAM. J. Math. Anal., Vol. 46(3) (2014), 1874–1892.

[20] Ursin, B., *Review of elastic and electromagnetic waves propagation in horizontally layered media*, Geophysics, Vol. 48 (1983), 1063–1081.

[21] Azeredo, M.M., *Modelagem matemática e computacional da propagação de ondas sísmicas em meios poroelásticos estratificados*, PhD thesis, LENEP, Universidade Estadual do Norte Fluminense Darcy Ribeiro, Campos dos Goytacazes, RJ, Brazil, 2013.

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Received 10.10.2017, Accepted 30.10.2017