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INVERSE PROBLEMS FOR EQUATIONS WITH A MEMORY

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Abstract Integro-differential equations of the electrodynamics with dispersion and viscoelasticity equations are considered in this paper. These equations differ from the usual equations of electrodynamics and elasticity by convolutions terms which lead to a dependence of solutions to these equations on a prehistory of a process. Hence, they have a special type of "memory". Then some new inverse problems occur. Along with parameters of a medium we need recover kernels of integral operators. Below we give a review of some results for inverse problems in this direction.

Key words: inverse problems, electrodynamic equations, viscoelasticity equations, uniqueness, stability estimates.

AMS Mathematics Subject Classification: 35C20, 35L15, 35R10, 45Q05.

1 Introduction

We consider equations of the electrodynamics and elasticity which contain integral convolution terms. Such type of equations describes in the electrodynamics processes with a dispersion while in the elasticity it describes an influence of a viscosity of a material. In both cases kernels of integral operators entering into the equations are usually unknown. The propagation of electromagnetic and elastic waves depends on these kernels. Thus, we come to the necessity of a consideration of some inverse problems related with the integro-differential equations. Since parameters of a medium (or coefficients of differential equations) are often also unknown, the inverse problems usually consist in a determination of some functions. A part of these functions depends on spatial variables only, while an other can depend on the time variable also.

Below we give a review of some results related to inverse problems for the electrodynamics and elasticity equations with "memory". In the next section we consider some statements of inverse problems for equations of the electrodynamics with a convolution term. They based on the author's papers [17, 18, 19, 25]. The typical inverse problem studied here consists in the following. The electric permeability $\varepsilon_0(x)$ is a given constant anywhere outside of a compact domain Ω with a smooth boundary $\partial\Omega$ and unknown inside Ω, a kernel $\varepsilon(x,t)$ is represented in the form $\varepsilon(x,t) = p(x)k(t)$, where $k(t)$ is a known function, while $p(x)$ is an unknown one, and support of it is contained in Ω. The unknown functions should be recovered from the trace of a solution to the Cauchy problem for the integro-differential electromagnetic equations given on $\partial\Omega$ for a finite time interval $[0,T]$. We give algorithms for solving these problems and stability estimates for the solutions.

In section 3 we consider two- and three-dimensional inverse problems for viscoelasticity equations. In general, these equations contain 3 medium parameters, density $\rho(x)$ and 2 Lamé moduli $\lambda(x)$ and $\mu(x)$, and two kernels $p(x,t)$ and $q(x,t)$. In the two-dimensional case only 3 functions, namely, $\rho(x)$, $\mu(x)$ and $p(x,t)$ enter in the equation. We consider a posing of a two-dimensional inverse problem with many observations in which a solution to the Cauchy problem with initial zero data depends on a parameter y that is a point of a concentrated force application belonging $\partial\Omega$. Then we assume that the functions $\rho(x)$, $\mu(x)$ are unknown inside Ω and are given positive constants outside of this domain. The function $p(x,t)$ is supposed to be represented as $p(x,t) = p_0(x)k(t)$, where $k(t)$ is given, while the support of $p_0(x)$ lies in Ω and $p_0(x)$ is unknown. We demonstrate that all three unknown functions can be uniquely found from the solutions to the direct problems given for all $(x,y) \in (\partial\Omega \times \partial\Omega)$ and for a finite time interval $[0,T]$. We also study a two-dimensional inverse problem with a single observation where a source is fixed. In the latter case $\rho(x)$ and $\mu(x)$ are suppose be given anywhere. For the three-dimensional case, we consider the inverse problem assuming that density $\rho(x)$ is given, and that $\rho(x)$, $\lambda(x)$, $\mu(x)$ and $p(x,t)$, $g(x,t)$ are infinitely differentiable functions of its variables. In the setting with many observations we proof that the functions $\lambda(x)$, $\mu(x)$ as well as $\partial^n p(x,t)/\partial t|_{t=0}$, $\partial^n q(x,t)/\partial t^n|_{t=0}$, $n = 0, 1, 2, \ldots$, are uniquely determine in Ω by the displacements vector given for all $(x,y) \in (\partial\Omega \times \partial\Omega)$ and for a finite time interval $[0,T]$.

In the Section 4, for a reader's convenience, we derive sufficient conditions of non-positivety of the Riemannian conformal metric. These conditions are used in Sections 2 and 3. Moreover, we also derive a sufficient condition for a boundary of a compact domain be a convex with respect to geodesics of the Riemannian metric.

1 Inverse problems for the dispersion electrodynamic equations

The propagation of electromagnetic waves in dispersion media is described by the equations

$$\begin{aligned} &\frac{\partial D}{\partial t} - \mathrm{rot} H(x,t) + j(x,t) = 0, \\ &\mu(x)\frac{\partial}{\partial t} H(x,t) + \mathrm{rot} E(x,t) = 0; \quad (x,t) \in \mathbb{R}^4, \end{aligned} \tag{2.1}$$

where

$$D = \varepsilon_0(x)E(x,t) + \int_{-\infty}^{t} \varepsilon(x,t-s)E(x,s)\,ds.$$

In these equations, $\varepsilon_0(x)$ is the dielectric permeability of the medium and the coefficient $\varepsilon(x,t)$ characterizes the medium dispersion. The convolution $k * E$ corresponds to a certain "memory" of the medium. In what follows, we consider the system of equations (2.1) under the zero initial conditions

$$(E,H)_{t<0} = 0, \tag{2.2}$$

We assume that the function $\varepsilon_0(x) \geq \varepsilon_{00} > 0$ is a positive and $\varepsilon(x,t)$ can be represented in the form

$$\varepsilon(x,t) = k(t)p(x),$$

where k(0) = 1 and k(t) is a known function. Suppose that the supports of the functions $\varepsilon_0(x) - 1$ and $p(x)$ are contained in a compact open domain $\Omega \in \mathbb{R}^3$ with smooth boundary $\partial\Omega$. We are interested in determining a pair of functions $\varepsilon_0(x)$ and $p(x)$ from a certain information about the solution of problem (2.1), (2.2). We give an exact setting of this inverse problem a little latter. Suppose that the Riemannian metric $d\tau = \varepsilon_0^{1/2}(x)|dx|$, $|dx| = (dx_1^2 + dx_2^2 + dx_3^2)^{1/2}$, has non-positive curvature in Ω. A sufficient condition for this is the inequality

$$\sum_{i,j=1}^{3} \frac{\partial^2 \ln \varepsilon_0(x)}{\partial x_i \partial x_j} \nu_i \nu_j \geq 0, \quad \forall : \nu = (\nu_1, \nu_2, \nu_3) \neq 0, \quad x \in \Omega.$$

Suppose that, in addition, the domain Ω is convex with respect to geodesics (see the section 4, where the both a sufficient condition on non-positivity and a convexity condition are given). Under these conditions, the metric is simple in the closed domain $\overline{\Omega}$; i.e., any two points x and y in $\overline{\Omega}$ can be joined by a unique geodesic. We denote the geodesic line joining points x and y by $\Gamma(x,y)$ and its Riemannian length by $\tau(x,y)$.

Below we consider two different posing of the inverse problems. First of them is related with many observations, while the other one with a single observation.

2.1 An inverse problem with many observations

We assume herewith that exterior current having the form of a moment dipole concentrated at a point $y \in \partial\Omega$ and having direction $j_0(y)$; i.e., $j = j_0(y)\delta(x-y,t)$. We also assume that $j_0(y) \neq 0$ and lies in the tangent direction to Ω at $y \in \partial\Omega$. Consider the following inverse problem. Let $\eta > 0$ be an arbitrary number. Suppose that the function $H(x,t,y)$ is given for all $(x,y) \in (\partial\Omega \times \partial\Omega)$ and all $t \leq \tau(x,y) + \eta$, i.e.,

$$H(x,t,y) = f(x,t,y), \quad (x,y) \in (\partial\Omega \times \partial\Omega), \quad t \leq \tau(x,y) + \eta. \tag{2.3}$$

It is required to determine $\varepsilon_0(x)$ and $p(x)$ in Ω from the function $f(x,t,y)$.

Below, following [19], we give some arguments which reduce this problem to simpler problems to be solved successively. In the case where $\varepsilon_0(x) \equiv 1$, $\varepsilon(x,t) \equiv 0$ the solution of problem (2.1), (2.2) has the form

$$\begin{aligned}
H(x,t) &= \operatorname{rot}\Big[\frac{j^0}{4\pi|x-y|}\,\delta(t-|x-y|)\Big] \\
&= \frac{j^0 \times \nu}{4\pi|x-y|}\Big[\delta'(t-|x-y|) + \frac{1}{|x-y|}\,\delta(t-|x-y|)\Big], \\
E(x,t) &= -\frac{\partial}{\partial t}\Big[\frac{j^0}{4\pi|x-y|}\,\delta(t-|x-y|)\Big] + \nabla \operatorname{div}\Big[\frac{j^0}{4\pi|x-y|}\,\theta_0(t-|x-y|)\Big] \\
&= \frac{1}{4\pi}\Big\{\frac{\nu(\nu\cdot j^0) - j^0}{|x-y|}\,\delta'(t-|x-y|) + \frac{3\nu(\nu\cdot j^0) - j^0}{|x-y|^2}\,\delta(t-|x-y|) \\
&\quad + \frac{3\nu(\nu\cdot j^0) - j^0}{|x-y|^3}\,\theta_0(t-|x-y|)\Big\},
\end{aligned} \tag{2.4}$$

where $\nu = (x-y)/|x-y|$ and $\theta_0(t)$ is the Heaviside function. In what follows, we assume that the point $y \in \partial\Omega$ is a variable parameter of the problem.

Lemma 2.1. *If the functions $\varepsilon_0(x)$ and $\varepsilon(x,t)$ satisfy the assumptions made above and are sufficiently smooth (say, of class $\mathbf{C}^\infty(\Omega \times \mathbb{R})$), then, for small $\eta > 0$, the solution of problem (2.1), (2.2) can be represented in a form similar to (2.4); namely,*

$$\begin{aligned} H(x,t,y) &= \alpha_H(x,y)\delta'(t-\tau(x,y)) + \beta_H(x,y)\delta(t-\tau(x,y)) + \hat{H}(x,t,y), \\ E(x,t,y) &= \alpha_E(x,y)\delta'(t-\tau(x,y)) + \beta_E(x,y)\delta(t-\tau(x,y)) + \hat{E}(x,t,y), \\ &\qquad t \le \tau(x,y) + \eta. \end{aligned} \tag{2.5}$$

Here, $\alpha_H(x,y)$, $\beta_H(x,y)$ and $\alpha_E(x,y)$, $\beta_E(x,y)$ are solutions of the equations

$$\begin{aligned} &(2\nabla\tau(x,y)\cdot\nabla + \Delta\tau(x,y) + \varepsilon_0(x)p(x))\alpha_H(x,y) - \nabla\varepsilon_0(x) \times \alpha_E(x,y) = 0, \\ &(2\nabla\tau(x,y)\cdot\nabla + \Delta\tau(x,y) + \varepsilon_0(x)p(x))\beta_H(x,y) - \Delta\alpha_H(x,y) \\ &+k'(0)\varepsilon_0(x)p(x)\alpha_H(x,y) - \nabla p(x) \times \alpha_E(x,y) - \nabla\varepsilon_0(x) \times \beta_E(x,y) = 0, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\varepsilon_0(x)\alpha_E(x,y) + \nabla\tau(x,y) \times \alpha_H(x,y) = 0, \\ &\varepsilon_0(x)\beta_E(x,y) + \varepsilon_0(x)p(x)\alpha_E(x,y) + \nabla\tau(x,y) \times \beta_H(x,y) \\ &\qquad -\mathrm{rot}\,\alpha_H(x,y) = 0, \end{aligned} \tag{2.7}$$

and satisfy the limit conditions

$$\begin{aligned} \lim_{x\to y}[\alpha_H(x,y)\tau(x,y)] &= \frac{j^0(y)\times\nu(y)}{4\pi}, \\ \lim_{x\to y}[\beta_H(x,y)\tau(x,y)] &= \frac{j^0(y)\times\nu(y)}{4\pi}, \end{aligned} \tag{2.8}$$

as x tends to y along the geodesic $\Gamma(x,y)$; here, $\nu(y)$ is the unit tangent vector to $\Gamma(x,y)$ at the point y. The functions $\hat{H}(x,t,y)$ and $\hat{E}(x,t,y)$ are certain functions of the variables (x,t) which are regular at $x \ne y$ and, moreover, satisfy the conditions $\hat{H}(x,t,y) = 0$ and $\hat{E}(x,t,y) = 0$ for all $t < \tau(x,y)$.

Below we briefly outline the proof of this lemma. It is convenient to construct a second order equation for function $H(x,t,y)$ and consider the system of this second order equation for $H(x,t,y)$ and a first order equation for $E(x,t,y)$. Substituting representation (2.5) into this system, we find relations (2.6) and (2.7). By virtue of the assumptions made above, in some neighborhood of $y \in \partial\Omega$, we have $\varepsilon_0(x) = 1$ and $p(x) = 0$. Therefore, for each fixed point $y \in \partial\Omega$, formula (2.4) uniquely determines the values of the functions $\alpha_H(x,y)$, $\beta_H(x,y)$ and $\alpha_E(x,y)$, $\beta_E(x,y)$ in some neighborhood of this point. This implies the limit relations (2.8). As a result, the values of the functions $\alpha_H(x,y)$ and $\beta_H(x,y)$ inside Ω are constructed, after eliminating $\alpha_E(x,y)$ and $\beta_E(x,y)$ from Eqs. (2.6) by using algebraic relations (2.7), along geodesics as solutions of ordinary differential equations, after which the functions $\alpha_E(x,y)$ and $\beta_E(x,y)$ are found in explicit form. The system of equations for $\hat{H}(x,t,y)$, $\hat{E}(x,t,y)$, which results from substituting representation (2.5), makes it possible to prove the regularity of these functions. It follows from (2.5) that

$$f(x,t,y) = \alpha_H(x,y)\delta'(t-\tau(x,y)) \\ +\beta_H(x,y)\delta(t-\tau(x,y)) + \hat{f}(x,t,y). \qquad (2.9)$$

Here, $\hat{f}(x,t,y)$ is the regular part of the function $f(x,t,y)$. Note that $\alpha_H(x,y) \neq 0$ for all $(x,y) \in (\partial\Omega \times \partial\Omega)$ and $x \neq y$ (see the corollary of Lemma 2.2 stated below). Let $x \in \partial\Omega$ and $y \in \partial\Omega$ be fix points. Consider the function $f(x,t,y)$ as a function of the variable t. This function identically vanishes at $t < \tau(x,y)$ and has nonzero singular part at a $t = \tau(x,y)$, because $\alpha_H(x,y) \neq 0$. Therefore, $\tau(x,y) = \sup\{\tau\}$, $\{\tau\} = \{\tau \in \mathbb{R} |\, u(x,t,y) \equiv 0, \text{ if } t < \tau\}$. Note that

$$\alpha_H(x,y) = \lim_{t\to\tau(x,y)+0} \int_{-\infty}^{t} (t-s)f(x,s,y)\,ds, \quad (x,y) \in (\partial\Omega \times \partial\Omega). \qquad (2.10)$$

Thus, the inverse problem stated above can be reformulated as follows: determine $\varepsilon_0(x)$ and $p(x)$ inside Ω from functions $\tau(x,y)$ and $\alpha_H(x,y)$ given for all $(x,y) \in (\partial\Omega \times \partial\Omega)$. This problem, in turn, reduces to the following two problems to be solved successively.

Problem 2.1. Determine $\varepsilon_0(x)$ inside Ω from a function $\tau(x,y)$ given for any pair of points x, y belonging to $\partial\Omega$.

Problem 2.2. For a known function $\varepsilon_0(x)$, determine $p(x)$ inside Ω from a function $\alpha_H(x,y)$ given at $(x,y) \in (\partial\Omega \times \partial\Omega)$.

Problem 2.1 is called *the inverse kinematic problem* and has been well studied (see, e.g., books [15, 16]). The stability of its solution was estimated in the case of a two-dimensional space by R. Mukhometov in [12] and in the case of a higher dimensional space, in [5, 6, 13]. Here, we consider Problem 2.2.

Lemma 2.2. *The function $|\alpha_H(x,y)|$ has the expression*

$$|\alpha_H(x,y)| = \frac{|j^0(y) \times \nu(y)|\sqrt{\det(\frac{\partial\zeta}{\partial x})}}{4\pi\tau(x,y)} \exp\Big(-\frac{1}{2}\int_{\Gamma(x,y)} p(\xi)\,d\tau\Big), \qquad (2.11)$$

in which $\zeta = (\zeta_1, \zeta_2, \zeta_3) = \nu(y)\tau(x,y)$ is the vector of Riemann coordinates of the point x and $\det(\frac{\partial\zeta}{\partial x})$ is the Jacobian of the transition from the Riemann coordinates to the Cartesian ones.

The proof of this lemma is fairly simple; we give it here, because formula (2.11) plays the fundamental role in the problem under consideration. Equalities (2.6) and (7) imply a differential equation for the function α_H. It has the form

$$(2\nabla\tau(x,y)\cdot\nabla + \Delta\tau(x,y) + \varepsilon_0(x)p(x))\alpha_H(x,y) \\ +\nabla\ln\varepsilon_0(x) \times (\nabla\tau(x,y) \times \alpha_H(x,y)) = 0. \qquad (2.12)$$

This equation readily implies $\alpha_H(x,y) \cdot \nabla\tau(x,y) = 0$. As is well known, the principal singularities of the vectors H and E are oriented in tangent directions to the front of the electromagnetic wave propagation. Using this fact and taking the inner products of both sides equality (2.12) and $\alpha_H(x,y)$, we obtain an equation for $|\alpha_H(x,y)|^2$ of the form

$$\nabla\tau(x,y)\cdot\nabla|\alpha_H(x,y)|^2+(\Delta\tau(x,y)+\varepsilon_0(x)p(x) \\ -\nabla\tau(x,y)\cdot\nabla\ln\varepsilon_0(x))|\alpha_H(x,y)|^2=0. \qquad (2.13)$$

Using formula (2.2.35) in [16] with $a_{ij}=\delta_{ij}/\varepsilon_0(x)$, we obtain

$$\Delta\tau(x,y)-\nabla\tau(x,y)\cdot\nabla\ln\varepsilon_0(x)=\varepsilon_0(x)\Big[\frac{2}{\tau(x,y)}-\frac{d}{d\tau}\ln\det\Big(\frac{\partial\zeta}{\partial x}\Big)\Big],$$

in which $\frac{d}{d\tau}$ denotes differentiation along the geodesic $\Gamma(x,y)$. Observing that $\nabla\tau(x,y)\cdot\nabla=\varepsilon_0(x)\frac{d}{d\tau}$, we find

$$\frac{d}{d\tau}\left(\frac{|\alpha_H(x,y)|^2\tau^2(x,y)}{\det\big(\frac{\partial\zeta}{\partial x}\big)}\exp\int\limits_{\Gamma(x,y)}p(\xi)\,d\tau\right)=0. \qquad (2.14)$$

Using the limit equality (2.8) for the function $\alpha_H(x,y)$, we obtain formula (2.11).

Corollary 2.1. *For all $(x,y)\in\partial\Omega\times\partial\Omega$ such that $x\neq y$, $\alpha_H(x,y)\neq 0$.*

Formula (2.11) implies the relation

$$\int\limits_{\Gamma(x,y)}p(\xi)\,d\tau=-2\ln\frac{4\pi\tau(x,y)|\alpha_H(x,y)|}{|j^0(y)\times\nu(y)|\sqrt{\det(\frac{\partial\zeta}{\partial x})}}\equiv g(x,y), \\ (x,y)\in(\partial\Omega\times\partial\Omega), \qquad (2.15)$$

which reduces Problem 2.2 under consideration to an integral geometry problem studied in [5, 6, 12, 14]. Below, we give a result related to the stability of its solution, following [16]. Suppose that a function $x=\chi(\xi)$ implements an one to one mapping of class $\mathbf{C}^2$ from the unit sphere S^2 centered at the origin to $\partial\Omega$ so that the positive orientation of $\partial\Omega$ corresponds to the positive orientation of S^2. Suppose also that θ and φ are the spherical angles of the point $\xi\in S^2$ and $\Upsilon=[0,\pi]\times[0,2\pi]$ is the range of variation of the variables θ and φ. Finally, suppose that points $x\in\partial\Omega$ and $y\in\partial\Omega$ are mapped to points $\xi(\theta_1,\varphi_1)$ and $\xi(\theta_2,\varphi_2)$, respectively. Then $x=\chi(\xi(\theta_1,\varphi_1))\equiv x(\theta_1,\varphi_1)$ and $y=\chi(\xi(\theta_2,\varphi_2))\equiv y(\theta_2,\varphi_2)$. We set $g(x(\theta_1,\varphi_1),y(\theta_2,\varphi_2))\equiv\bar g(\theta_1,\varphi_1,\theta_2,\varphi_2)$. Then relation (2.15) can be written in the form

$$\int\limits_{\Gamma(x(\theta_1,\varphi_1),y(\theta_2,\varphi_2))}p(\xi)\,d\tau=\bar g(\theta_1,\varphi_1,\theta_2,\varphi_2), \\ (\theta_1,\varphi_1)\in\Upsilon,\quad(\theta_2,\varphi_2)\in\Upsilon. \qquad (2.16)$$

We also set $\tau(x(\theta_1,\varphi_1),y(\theta_2,\varphi_2))\equiv\bar\tau(\theta_1,\varphi_1,\theta_2,\varphi_2)$, by $I(\bar g,\bar\tau)$ we denote the determinant

$$I(\bar g,\bar\tau)=\det\begin{pmatrix}0 & \bar g_{\theta_1} & \bar g_{\varphi_1}\\ \bar g_{\theta_2} & \bar\tau_{\theta_1\theta_2} & \bar\tau_{\varphi_1\theta_2}\\ \bar g_{\varphi_2} & \bar\tau_{\theta_1\varphi_2} & \bar\tau_{\varphi_1\varphi_2}\end{pmatrix}.$$

Theorem 2.1. *If $\partial\Omega \in \mathbf{C}^2$, $\varepsilon_0(x) \in \mathbf{C}^2(\Omega \cup \partial\Omega)$, $p(x) \in \mathbf{C}^1(\Omega \cup \partial\Omega)$ and the family of geodesics is simple in Ω, then the stability of the solution is estimated as*

$$\int\limits_{\Omega} p^2(x)\varepsilon_0^{3/2}(x)\,dx \le -\frac{1}{8\pi} \int\limits_{\Upsilon\times\Upsilon} I(\bar{g},\bar{\tau})\,d\theta_1 d\varphi_1 d\theta_2 d\varphi_2. \tag{2.17}$$

2.2 An inverse problem with a single observation

As the exterior current $j(x,t)$ we take the function $j = j_0\delta(t)\delta(x_1)$ with $j_0 = (0,0,1) = \mathbf{e_3}$ and the Dirac-function $\delta(t)$. Henceforth $\mathbf{e_k}$ for $k = 1,2,3$ stands for the unit vector along the axis x_k. Suppose that the plane $x_1 = 0$, on which the function $\delta(x_1)$ is localized, lies outside the closure of Ω. Assume for definiteness that Ω lies in the half-space $\mathbb{R}^3_d = \{x|\,x_1 \ge d\}$ for some $d > 0$. Seeking some technical simplifications in studying the inverse problem, we assume also that $\varepsilon_0(x) = 1$ outside $\mathbb{R}^3_d$

Introduce a function $\tau(x)$ as the solution to the Cauchy problem

$$|\nabla\tau(x)|^2 = \varepsilon_0(x); \quad \tau|_{x_1=0} = 0. \tag{2.18}$$

Consider some cylindrical domain $G = G(T) = \{(x,t)|\,x \in \Omega, \tau(x) < t < T + \tau(x)\}$, where T is some positive number. Denote by $S = S(T) = \{(x,t) \in (\partial\Omega \times \mathbb{R})|\,\tau(x) < t < T + \tau(x)\}$ its lateral surface, and by $\Sigma = \Sigma(T)$, its lower base $\Sigma(T) = \{(x,t)|\,x \in \Omega, t = \tau(x) + 0\}$.

The arguments of Lemma 2.3 (see below) imply that under certain assumptions on the coefficients of (2.1) we can express the functions $E(x,t)$ and $H(x,t)$, which constitute a solution to (2.1), (2.2), as the sums of 2 certain singular functions supported on the characteristic surface $t = \tau(x)$ and regular functions $\bar{E}(x,t)$ and $\bar{H}(x,t)$ supported on $\{(x,t)|\,t \ge \tau(x)\}$; namely,

$$E(x,t) = \alpha_E(x)\delta(t-\tau(x)) + \bar{E}(x,t), \quad H(x,t) = \alpha_H(x)\delta(t-\tau(x)) + \bar{H}(x,t).$$

We assume here that $\varepsilon_0(x)$ is known function anywhere. In order to find $p(x)$ on Ω, we assume that for the solution to (2.1), (2.2) we know on S the values of the magnetic field H and its normal derivative, as well as the coefficient of the singular part of $H(x,t)$:

$$H|_S = g(x,t), \quad \frac{\partial H}{\partial n}\Big|_S = h(x,t), \quad \alpha_H|_{\partial\Omega} = \alpha'_H(x). \tag{2.19}$$

Problem 2.3. Given the functions $g(x,t)$ and $h(x,t)$, and $\alpha'_H(x)$, we are required to find $p(x)$ on Ω.

For fixed numbers $q_0 > 0$ and $d > 0$ denote by $\Lambda(q_0, d)$ the set of functions (ε_0, k, p) satisfying the following conditions:

(1) $\operatorname{supp} p(x) \subset \Omega$, $\operatorname{supp}(\varepsilon_0(x) - 1) \subset \Omega$,

(2) $\|p\|_{\mathbf{C}^8(\mathbb{R}^3)} \le q_0$, $\|\varepsilon_0 - 1\|_{\mathbf{C}^{10}(\mathbb{R}^3)} \le q_0$ and $\|k-1\|_{\mathbf{C}^4[0,\infty)} \le q_0$.

The solution to the direct problem satisfies

Lemma 2.3. *For every $T_0 > 0$ there is a positive number $q_0^* = q_0^*(T_0)$ such that for every $(\varepsilon_0, k, p) \in \Lambda(q_0, d)$, $q_0 \le q_0^*$, we can express the solution (E, H) to (2.1)-(2.2) in the domain*

$$K(T_0) = \{(x,t) |\, \tau(x) \le T_0,\, 0 \le t \le T_0 - \tau(x)\}$$

as

$$\begin{aligned} E(x,t) &= \alpha_E(x)\delta(t-\tau(x)) + \beta_E(x)\theta_0(t-\tau(x)) \\ &\quad + \gamma_E(x)\theta_1(t-\tau(x) + \hat{E}(x,t), \\ H(x,t) &= \alpha_H(x)\delta(t-\tau(x)) + \beta_H(x)\theta_0(t-\tau(x)) \\ &\quad + \gamma_H(x)\theta_1(t-\tau(x) + \hat{H}(x,t), \end{aligned} \tag{2.20}$$

where $\delta(t)$ is the Dirac delta-function and $\theta_0(t)$ is the Heaviside function: $\theta_0(t) = 1$ for $t \ge 0$ and $\theta_0(t) = 0$ for $t < 0$, while $\theta_1(t) = t\,\theta_0(t)$. The coefficients α_E, β_E, γ_E and α_H, β_H, γ_H are smooth functions of the spatial variable x. Furthermore,

$$\begin{aligned} \alpha_E &\in \mathbf{C}^8(D(T_0)), \quad \alpha_H \in \mathbf{C}^8(D^+(T_0) \cup D^-(T_0)), \\ \beta_E &\in \mathbf{C}^6(D(T_0)), \quad \beta_H \in \mathbf{C}^6(D^+(T_0) \cup D^-(T_0)), \\ \gamma_E &\in \mathbf{C}^4(D(T_0)), \quad \gamma_H \in \mathbf{C}^4(D^+(T_0) \cup D^-(T_0)), \\ D(T_0) = \{x | \tau(x) &\le T_0\}, \quad D^+(T_0) = D(T_0) \cap \{x|\, x_1 > 0\}, \\ & \qquad D^-(T_0) = D(T_0) \cap \{x|\, x_1 < 0\}, \end{aligned}$$

while the functions $\hat{E}(x,t)$ and $\hat{H}(x,t)$ along with their partial derivatives with respect to t are of class $\mathbf{H}^3(\Sigma(T_0, t_0))$, $\Sigma(T_0, t_0) = K(T_0) \cap \{(x,t)|\, t = t_0\}$, $t_0 \in (0, T_0]$, and vanish identically for $t \le \tau(x)$. Moreover, there exists a positive constant C such that for all $q_0 \le q_0^*$ we have

$$\begin{aligned} |\alpha_E(x) - \alpha_E^0(x)| \le Cq_0, \quad |\beta_E(x)| \le Cq_0, \quad |\gamma_E(x)| \le Cq_0, \quad x \in D(T_0), \\ |\hat{E}(x,t)| \le Cq_0, \quad |\hat{E}_t(x,t)| \le Cq_0, \quad (x,t) \in K(T_0), \end{aligned} \tag{2.21}$$

$$\begin{aligned} |\alpha_H(x) - \alpha_H^0(x)| \le Cq_0, \quad |\beta_H(x)| \le Cq_0, \quad |\gamma_H(x)| \le Cq_0, \quad x \in D(T_0), \\ |\hat{H}(x,t)| \le Cq_0, \quad |\hat{H}_t(x,t)| \le Cq_0, \quad (x,t) \in K(T_0). \end{aligned} \tag{2.22}$$

Here, $\alpha_E^0(x) = -\mathbf{e_3}/2$ and $\alpha_H^0(x) = (\mathbf{e_2}/2)\mathrm{sign}(x_1)$.

The main result here is the following theorem on the stability of the solution to the inverse problem 2.3.

Theorem 2.2. *Suppose that $(\varepsilon_0, k, p_i) \in \Lambda(q_0, d)$, $i = 1, 2$, while $g^i(x,t)$, $h^i(x,t)$ and $\alpha_H^i|_{\partial\Omega}(x)$ constitute the Cauchy data corresponding to the solution to (2.1), (2.2) for $p = p_i(x)$. In addition, suppose that Ω lies in a Riemannian ball of radius ρ and*

$$T > 4\rho. \tag{2.23}$$

Then there are positive numbers q_0^* and C such that for all $q_0 \le q_0^*$ we have

$$\begin{aligned} \|p_1 - p_2\|^2_{\mathbf{H}^1(\Omega)} \;\le\; & C\Big(\|g_t^1 - g_t^2\|^2_{\mathbf{H}^1(S)} + \|h_t^1 - h_t^2\|^2_{\mathbf{L}^2(S)} \\ & + \|\alpha_H^1 - \alpha_H^2\|^2_{\mathbf{H}^2(\partial\Omega)} + \|\beta_H^1 - \beta_H^2\|^2_{\mathbf{H}^2(\partial\Omega)} \Big). \end{aligned} \tag{2.24}$$

Here $\beta_H^i(x) = g^i(x, \tau(x) + 0)$, $i = 1, 2$.

Theorem 2.2 implies the uniqueness theorem

Theorem 2.3. *In the hypotheses of Theorem 2.2, suppose that the Cauchy data corresponding to two solutions to (2.1), (2.2) for $p = p_i$, $i = 1, 2$, coincide: $g^1 = g^2$, $h^1 = h^2$, and $\alpha^1_H|_{\partial\Omega} = \alpha^2_H|_{\partial\Omega}$. Then there is a positive number q^*_0 such that for $q_0 \le q^*_0$ we have $p_1(x) = p_2(x)$ in Ω.*

Proofs of the Lemma 2.3 and Theorem 2.2 are given in [19].

3 Inverse problems for the viscoelasticity equations

Wave propagation in modern composite materials is described by the integrodifferential equation

$$\rho(x)\, u_{tt} - Lu = F, \tag{3.1}$$

where $u = (u_1, u_2, u_3)$ is the elastic displacement vector, $\rho(x)$ is the density of the medium, $x \in \mathbb{R}^3$, $F = (F_1, F_2, F_3)$ is the force vector, and the operator $L = (L_1, L_2, L_3,)$ is defined by the equalities

$$L_i u = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(u)}{\partial x_j}, \quad i = 1, 2, 3,$$

$$\sigma_{ij}(u) = \lambda(x)\delta_{ij}\mathrm{div} u(x,t) + \mu(x)\left(\frac{\partial u_i(x,t)}{\partial x_j} + \frac{\partial u_j(x,t)}{\partial x_i}\right)$$
$$+ \int\limits_{-\infty}^{t} \left[p(x, t-s)\delta_{ij}\mathrm{div} u(x,s) + q(x,t-s)\left(\frac{\partial u_i(x,s)}{\partial x_j} + \frac{\partial u_j(x,s)}{\partial x_i}\right)\right] ds,$$
$$i, j = 1, 2, 3.$$

In these equalities δ_{ij} is the Kronecker symbol, while $\lambda(x)$ and $\mu(x)$ are the elasticity moduli, and the functions $p(x,t)$ and $q(x,t)$ characterize the viscosity of the medium. In what follows, we assume that $\lambda(x) + \mu(x) > 0$, $\mu(x) > 0$ and $\rho(x) > 0$.

Let
$$c_p(x) = \left(\frac{\lambda(x) + 2\mu(x)}{\rho(x)}\right)^{1/2}, \quad c_s(x) = \left(\frac{\mu(x)}{\rho(x)}\right)^{1/2}$$
be the speeds of the longitudinal and transverse waves respectively and let $\tau_p(x,y)$ and $\tau_s(x,y)$ be the geodesic distance corresponding the Riemannian metrics

$$d\tau_p = \frac{|dx|}{c_p(x)}, \quad d\tau_s = \frac{|dx|}{c_s(x)}, \quad |dx| = \sqrt{dx_1^2 + dx_2^2 + dx_3^2}.$$

Suppose that both Riemannian metrics are simple, i.e., that every pair of points x and y can be joined by a unique geodesic $\Gamma_p(x,y)$ and $\Gamma_s(x,y)$. In what follows, we will be interested in the problem of determining the functions $\rho(x)$, $\lambda(x)$, $\mu(x)$, $p(x,t)$ and

$q(x,t)$ from a given information about the family of solutions to some direct problems for (3.1). The inverse problems of determining the kernels $p(x,t)$ and $q(x,t)$ in (3.1) (or only the kernel $p(x,t)$ in the case when the system of equalities is replaced by a single scalar equation) under the assumption that $p(x,t)$ and $q(x,t)$ are representable in the forms $p(x,t) = k_1(t)p_0(x)$ and $q(x,t) = k_2(t)q_0(x)$, where $k_1(t)$ and $k_2(t)$ are given functions, while $p_0(x)$ and $q_0(x)$ are unknown functions whose support lies in some compact domain were studied earlier in [8, 10, 28]. For the first time a two-dimensional inverse problem of recovering kernel $p(x,t)$ of the forms $p(x,t) = k_1(t)p_0(x)$ under the assumption $\rho \equiv 1$ was considered in [8]. In this paper, it was assumed that two solutions of the Cauchy problem related to a scalar equation, that corresponding 2D case, with two different and nonzero initial data were known on boundary of a compact domain for a sufficiently large time interval $[0,T]$. The Hölder type stability estimate was found for this inverse problem. The similar inverse problem with a single observation was studied in [28]. In [10] stability estimates were found for a solution to the inverse problem of determining two kernels in the system of elasticity equations with the known density and elasticity modulus. In these cited above tree papers the method of Carleman estimates, introduced for inverse problems by A. Bukhgeim and M. Klibanov [9], was used. Unfortunately, an inverse problem with nonzero initial data can not be use in practical applications because if we can not produce measurements inside a medium, we do not able measure and nonzero initial data. To avoid this, we consider below equations (3.1) with zero initial date:

$$u|_{t<0} \equiv 0 \tag{3.2}$$

and study some inverse problems for equations (3.1), (3.2) in 2D and 3D cases with many or single obsevations following [20, 21, 22, 23, 24, 26].

3.1 2D inverse problems

Consider the equation

$$Lu \equiv \rho(x)u_{tt}(x,t) - \operatorname{div}\Big[\mu(x)\nabla u(x,t) + \int_{-\infty}^{t} p(x,t-s)\nabla u(x,s)\,ds\Big] = F(x,t) \tag{3.3}$$

for the function $u = u(x,t)$, $x = (x_1,x_2)$, $(x,t) \in \mathbb{R}^3$. The function $p(x,t)$ characterizes the viscosity of the medium, and the integral operator describes the influence of the prehistory on the process of propagation of elastic waves which is caused by an applied given force $F(x,t)$. Equation (3.3) appears in the theory of viscoelastic bodies whose properties do not depend on x_3. Then the third component of the displacement vector $u_3 = u(x,t)$ satisfies to equation (3.3). Suppose that the $u(x,t)$ is a solution to (3.3) satisfying the initial condition

$$u|_{t<0} = 0, \quad x \in \mathbb{R}^2. \tag{3.4}$$

Given $\rho(x)$, $\mu(x)$, $p(x,t)$ and $F(x,t)$, problem (3.3), (3.4) of determining $u(x,t)$ is well posed in suitable function spaces. Call it the *direct problem*. In what follows, we will be interested in the inverse problem of determining $\rho(x)$, $\mu(x)$ and $p(x,t)$ from a given information about the family of solutions to direct problems. Assume that $p(x,t)$ has a special structure; namely, $p(x,t)$ is representable as the product of two functions

$$p(x,t) = k(t)p_0(x), \tag{3.5}$$

in which $k(t) \in \mathbf{C}^2[0,\infty)$ is defined and satisfies the equation $k(0) = 1$, whereas $p_0(x)$ is unknown. Furthermore, assume that $\rho(x)$ and $\mu(x)$ are positive everywhere in $\mathbb{R}^2$ and different from given positive constants ρ_0 and μ_0 respectively only inside the unit disk $D = \{x \in \mathbb{R}^2 | \, |x| < 1\}$. More exactly, assume that

$$\operatorname{supp}\big((\rho(x) - \rho_0), (\mu(x) - \mu_0), p_0(x)\big) \subset D. \tag{3.6}$$

In the sequel, we will also make some assumptions about the smoothness of $\rho(x)$, $\mu(x)$ and $p_0(x)$ as well as the regularity of the field of geodesics connected with (3.3).

3.1.1 An inverse problem with many observations

Let $F(x,t)$ have the form

$$F(x,t) = \delta(x \cdot \nu(\psi) - 1)\delta(t), \quad \nu(\psi) = (\cos\psi, \sin\psi), \quad \psi \in [0, 2\pi], \tag{3.7}$$

where $\delta(t)$ is the Dirac delta-function and ψ is a variable parameter of the problem. In this case we denote the solution to the direct problem (3.3), (3.4), (3.7) by $u(x,t,\psi)$. Denote the boundary of the unit disk D by $\partial D = \{x \in \mathbb{R}^2 | \, |x| = 1\}$ and put

$$\partial_+ D = \{x \in \partial D | \, x \cdot \nu(\psi) > 0\}, \quad \partial_- D = \{x \in \partial D | \, x \cdot \nu(\psi) < 0\}.$$

Inverse problem 3.1. Suppose that, for a sufficiently large number $T > 0$ (see (3.12) below) $u(x,t,\psi)$ is known for all $(x,t,\psi) \in B(T) = \{x \in \partial_- D, t \in [0,T], \psi \in [0,2\pi]\}$:

$$u(x,t,\psi) = f(x,t,\psi), \; (x,t,\psi) \in B(T). \tag{3.8}$$

Given $f(x,t,\psi)$, find $\rho(x)$, $\mu(x)$ and $p_0(x)$ in D.

Denote by $c(x) = c_s(x) = \mu(x)/\rho_0(x)$ the propagation speed for shear waves. Let $\tau(x,\psi)$ be the solution to the Cauchy problem

$$|\nabla_x \tau(x,\psi)|^2 = \frac{1}{c^2(x)}; \quad \tau|_{x\cdot\nu(\psi)=1} = 0, \tag{3.9}$$

i.e., $\tau(x,\psi)$ is the distance between a point x and the straight line $\xi \cdot \nu(\psi) = 1$ on the plane of the variables $(\xi_1, \xi_2) = \xi$ in the Riemannian metric $d\tau = \sqrt{(d\xi_1^2 + d\xi_2^2)}/c(\xi)$. Suppose that the metric is simple in D. Denote by $\mathcal{Q}(m, M)$ the set of the functions $(\rho(x), \mu(x), p_0(x))$ satisfying the following conditions for fixed positive m, M:

1) $0 < m \le \rho(x) \le M$, $0 < m \le \mu(x) \le M$ for all $x \in \mathbb{R}^2$,
2) $\operatorname{supp}(\rho(x) - \rho_0, \mu(x) - \mu_0, p_0(x)) \subset D$,
3) $\rho(x) \in \mathbf{C}^4(D)$, $\mu(x) \in \mathbf{C}^4(D)$, $p_0(x) \in \mathbf{C}^2(D)$.

The equation $t = \tau(x, \psi)$ defines the front of a wave propagating from the source given by (3.7). In a neighborhood of this front, the solution to (3.3), (3.4), (3.7) has the form

$$\begin{aligned} u(x,t,\psi) &= A(x,\psi)\theta_0(t-\tau(x,\psi)) \\ &\quad +B(x,\psi)\theta_1(t-\tau(x,\psi)) + v(x,t,\psi), \end{aligned} \tag{3.10}$$

where $\theta_0(t)$ is the Heaviside function, the function $\theta_1(t)$ is defined by the equality $\theta_1(t) = t\theta_0(t)$, and $v(x,t,\psi)$ is an infinitesimal of higher order than $(t - \tau(x,\psi))$ and $v(x,t,\psi)$ vanishes for $t < \tau(x,\psi)$. In whole, the fact is known for hyperbolic equations; in application to the specific problem, it was used and established in [20, 23] for $\rho(x) = 1$. For getting relations for determining A, B, and $v(x,t,\psi)$, we must insert (3.10) in (3.3) and equate to zero the coefficients at $\delta'(t-\tau(x,\psi))$ and $\delta(t-\tau(x,\psi))$. Here we have used (3.10) to reduce the initial inverse problem to another problem. As a consequence of (3.10) the function $f(x,t,\psi)$, defined in the inverse problem, admits the representation

$$\begin{aligned} f(x,t,\psi) &= A(x,\psi)\theta_0(t-\tau(x,\psi)) \\ &\quad +B(x,\psi)\theta_1(t-\tau(x,\psi)) + \hat{f}(x,t,\psi), \end{aligned} \tag{3.11}$$

in which $\hat{f}(x,t,\psi)$ is an infinitesimal of a higher order than $(t-\tau(x,\psi))$ and $\hat{f}(x,t,\psi)$ vanishes for $t < \tau(x,\psi)$. Suppose that

$$T > \sup_{\psi\in[0,2\pi]} \sup_{x\in\partial_- D} \tau(x,\psi). \tag{3.12}$$

Then $f(x,t,\psi)$ defines $\tau(x,\psi)$, $A(x,\psi)$ and $B(x,\psi)$ uniquely for all $x \in \partial_- D$ and $\psi \in [0, 2\pi]$. The formulas for their calculation look as

$$\begin{gathered} \tau(x,\psi) = \sup\{\tau\}, \quad \{\tau\} = \{\tau \in \mathbb{R} |\, f(x,t,\psi) \equiv 0, \text{if}\, t < \tau\}, \\ A(x,\psi) = \lim_{t\to\tau(x,\psi)+0} f(x,t,\psi), \quad B(x,\psi) = \lim_{t\to\tau(x,\psi)+0} f_t(x,t,\psi). \end{gathered} \tag{3.13}$$

In this connection, we can consider the reduced statement of the inverse problem: The functions $\tau(x,\psi)$, $A(x,\psi)$ and $B(x,\psi)$ are known for all $(x,\psi) \in \Upsilon =: \{(x,\psi) | x \in \partial_- D, \psi \in [0, 2\pi]\}$; find $\rho(x)$, $\mu(x)$ and $p_0(x)$ in D. Below, we will see that this problem decomposes into the three consecutively solved problems:

(1) find $c(x) = \sqrt{\mu(x)/\rho(x)}$ from $\tau(x,\psi)$,

(2) given $c(x)$ and $A(x,\psi)$, find $\hat{p}(x) = p_0(x)/\mu(x)$;

(3) given $c(x)$, $\hat{p}(x)$ and $A(x,\psi)$, $B(x,\psi)$, find $\rho(x)$.

The first problem is the familiar inverse kinematic seismic problem. Two other problems lead to a linear problem of integral geometry on the family of geodesics of the Riemannian metric $d\tau = |dx|/c(x)$. Moreover, for determining $\rho(x)$ in D in the third problem the necessity appears of solving a boundary value problem for some second-order linear equation of elliptic type with Cauchy data on ∂D. A solution to each of these problems is unique (see below). This implies the uniqueness theorem:

Theorem 3.1. *Suppose that $(\rho_k, \mu_{0k}, p_k) \in \mathcal{Q}(m, M)$ and $f_k(x,t,\psi)$are the data (3.8) corresponding to solutions to (3.3), (3.4) for $\rho = \rho_k$, $\mu = \mu_k$, $p = p_{0k}$, $k = 1, 2$, and condition (3.12) is fulfilled. If, moreover, $f_1(x,t,\psi) = f_2(x,t,\psi)$; then $\rho_1(x) = \rho_2(x)$, $\mu_1(x) = \mu_2(x)$ and $p_{01}(x) = p_{02}(x)$ for all $x \in D$.*

Below following [24], we give some formulas for finding the amplitude coefficients $A(x,\psi)$ and $B(x,\psi)$. Basing on these formulas, we justify the decomposition of the initial problem into the above three simpler problems and give stability estimates for a solution to the inverse kinematic problem and the problem of integral geometry from which the uniqueness theorems for solutions to the corresponding problems follow as easy consequences. In fact, the contents of this subsection defines an algorithmic procedure for constructing the solution to the initial inverse problem.

1. At first, we give formulas for finding $A(x,\psi)$ and $B(x,\psi)$. The assumptions about the coefficients of (3.3) imply that, for every triple of the functions $\rho(x)$, $\mu(x)$ and $p_0(x)$, there exists $d\in(0,1)$ such that, in $D_d=\{x\in D|\,|x|>1-d\}$, the coefficients are constant: $\rho(x)=\rho_0$, $\mu(x)=\mu_0$, $p_0(x)=0$. Let $G_d=\{x\in\mathbb{R}^2|\,x\cdot\nu(\psi)>1-d\}$. Then (see [24])

$$A(x,\psi)=A_0=:\frac{c_0}{2\mu_0},\quad B(x,\psi)=0,\quad x\in G_d.$$

Moreover, in $G=\{x\in\mathbb{R}^2|\,x\cdot\nu(\psi)<1\}$ formula for $A(x,\psi)$ has the form

$$\begin{aligned} A(x,\psi)&=\frac{A_0\sqrt{\rho_0}}{\sqrt{\rho(x)}}\exp\left(a(x)\right),\\ a(x,\psi)&=\frac{1}{2}\int\limits_{\Gamma(x,\psi)}\Big[\hat{p}(\xi)-\operatorname{div}\left(c^2(\xi)\nabla\tau(\xi,\psi)\right)\Big]\,d\tau,\quad x\in G, \end{aligned}\tag{3.14}$$

where $\Gamma(x,\psi)$ is the geodesic passing on the plane of ξ_1,ξ_2 through a point x and orthogonal to the straight line $\xi\cdot\nu(\psi)=1$, while $d\tau$ is the element of the Riemannian length. Here $A(x,\psi)\in\mathbf{C}^2(D)$ for every fixed $\psi\in[0,2\pi]$. Moreover, $A(x,\psi)$ is positive everywhere. The function $B(x,\psi)$ is calculated in G by the formula

$$B(x,\psi)=-\frac{1}{2}A(x,\psi)\int\limits_{\Gamma(x,\psi)}\Big[\hat{C}(\xi,\psi))+h(\xi)\Big]\,d\tau.\tag{3.15}$$

where

$$\begin{aligned} \hat{C}(x,\psi)=\hat{p}(x)\big[\hat{p}(x)-\operatorname{div}\big(c^2(x)\nabla\tau(x,\psi)\big)-k'(0)\big]&\\ +\operatorname{div}[c^2(x)\hat{p}(x)\nabla\tau(x,\psi)]-\operatorname{div}[c^2(x)\nabla a(x,\psi)]&\\ -c^2(x)|\nabla a(x,\psi)|^2,&\qquad(3.16)\\ h(x)=\operatorname{div}[c^2(x)\nabla q(x)]+c^2(x)|\nabla q(x)|^2&\\ =\operatorname{div}[c^2(x)\nabla(\sqrt{\rho(x)})]/\sqrt{\rho(x)}.&\qquad(3.17) \end{aligned}$$

and $q(x)=\ln\sqrt{\rho(x)}$.

2. Decomposition of the Initial Inverse Problem.

The formulas given above and (3.9) imply the decomposition of the initial inverse Problem 3.1 into three separate problems to be solved consecutively. Write down their more detailed statements.

Problem 3.1.1. Find $c(x)=\sqrt{\mu(x)/\rho(x)}$ inside D from $\tau(x,\psi)$, $(x,\psi)\in\Upsilon$, $\Upsilon=:\{(x,\psi)|\,x\in\partial_-D,\psi\in[0,2\pi]\}$. The function $\tau(x,\psi)$ is a solution to (3.9). This problem is nonlinear. The stability estimate to the problem given below, imply the unique solution. Let $(\rho_k, \mu_k, p_{0k}) \in \mathcal{Q}(m, M)$, $k = 1, 2$. Denote $c_k(x) = \sqrt{\mu_k(x)/\rho_k(x)}$, $k = 1, 2$, and introduce

$$\tilde{c}(x) = c_1(x) - c_2(x), \quad \tilde{\tau}(x, \psi) = \tau_1(x, \psi) - \tau_2(x, \psi).$$

Then [24]:

$$\|\tilde{c}\|_{\mathbf{L}^2(D)} \leq C\|\tilde{\tau}\|_{\mathbf{H}^1(\Upsilon)}, \tag{3.18}$$

where $C = C(m, M)$ is a positive constant.

Once $c(x)$ has been found, $\Gamma(x, \psi)$ and $\tau(x, \psi)$ become known for all $x \in D$. Now, consider the following problem:

Problem 3.1.2. Given $c(x)$ and $A(x, \psi)$ for $(x, \psi) \in \Upsilon$, find $\hat{p}(x) = p_0(x)/\mu(x)$.

Use (3.14). Put $x \in \partial_- D$. Then $\rho(x) = \rho_0$ and

$$\int\limits_{\Gamma(x,\psi)} \hat{p}(\xi)\, d\tau = g_1(x, \psi), \quad (x, \psi) \in \Upsilon, \tag{3.19}$$

where the right-hand side is the known function

$$g_1(x, \psi) = 2(\ln A(x, \psi) - \ln A_0) + \int\limits_{\Gamma(x,\psi)} \operatorname{div}\left(c^2(\xi)\nabla\tau(\xi, \psi)\right) d\tau.$$

Thus, the problem is reduced to a linear problem of integral geometry: find the integrand from its integrals along a family of known geodesics. An estimate of the stability of this problem is given by the formula (see [24]):

$$\int\limits_D \hat{p}^2(x)\, dx \leq -\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi} \frac{\partial g_1(x(\varphi), \psi)}{\partial \psi}\frac{\partial g_1(x(\varphi), \psi)}{\partial \varphi}\, d\varphi\, d\psi, \tag{3.20}$$

where $x(\varphi) = (cos\varphi, \sin\varphi)$. Since the problem is linear and, by (3.20), $\hat{p} = 0$ corresponds to $g_1(x(\varphi), \psi) = 0$, a solution to the problem of integral geometry is unique.

Problem 3.1.3. Given $c(x)$, $\hat{p}(x)$ and $A(x, \psi)$, $B(x, \psi)$ for $(x, \psi) \in \Upsilon$, find $\rho(x)$ inside D.

In this case use (3.15) on putting $x \in \partial_- D$ therein. We obtain

$$\int\limits_{\Gamma(x,\psi)} h(\xi)\, d\tau = g_2(x, \psi), \quad (x, \psi) \in \Upsilon, \tag{3.21}$$

in which

$$g_2(x, \psi) = -2\frac{B(x, \psi)}{A(x, \psi)} - \int\limits_{\Gamma(x,\psi)} \hat{C}(\xi, \psi))\, d\tau,$$

and $\hat{C}(x, \psi))$ is defined by (3.16) and is known for all values of variables. This implies that, finding $h(x)$ in D, we come to an identical problem of integral geometry as before.

If it is solved and $h(x)$ is found then (3.17) implies a linear equation of elliptic type for $\sqrt{\rho(x)}$:

$$\operatorname{div}[c^2(x)\nabla(\sqrt{\rho(x)})] - h(x)\sqrt{\rho(x)} = 0, \quad x \in D.$$

By hypothesis, $\rho(x)$ on the boundary of D satisfies the boundary condition

$$\rho(x) = \rho_0, \quad \nabla\rho(x) = 0, \quad x \in \partial D.$$

It is known that a solution to an elliptic equation with Cauchy data on the boundary of a compact domain is unique. Therefore, a solution to Problem 3.1.3 is also unique.

Thus, the totality of the facts about the uniqueness of a solution to all above problems implies the uniqueness theorem for a solution to the initial boundary value problem.

3.1.2 An inverse problem with single observation

Assume that the density of the medium is the constant $\rho = 1$ and $\mu(x)$ is a given positive function $\mu(x) \geq \mu_0 > 0$, and $\mu(x) = 1$ outside Ω. We assume also that $p(x,t)$ can be represented in the form

$$p(x,t) = k(t)\mu(x)p_0(x), \tag{3.22}$$

where $k(t)$ is a given function such that $k(0) = 1$, while $p_0(x)$ is the unknown function. We suppose that the support of $p_0(x)$ lies in an open compact domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Consider a solution of (3.3) with function $F(x,t)$ of the form

$$F(x,t) = \delta(x_1)\delta(t),$$

and zero initial data (3.4). Assume that the trace of the solution to problem (3.3), (3.4) and its normal derivative are known on some finite piece $S \subset (\partial\Omega \times \mathbb{R})$ of the lateral boundary of $Q = \Omega \times \mathbb{R}$,

$$u|_S = g(x,t), \quad \left.\frac{\partial u}{\partial n}\right|_S = h(x,t). \tag{3.23}$$

Below we give a more exact description of S.

Problem 3.2.1. Given $g(x,t)$ and $h(x,t)$ find $p_0(x)$in Ω.

We assume that the line $x_1 = 0$ has no an intersection with Ω. Let, for a definiteness, Ω lies in the half-plain $x_1 > d$ for some $d > 0$.

Note the physical sense of the functions g and h which present data of the inverse problem. The function $g(x,t)$ is the thirs component of the displacement vector, while $h(x,t)$ is expressed via the normal stress σ_{3n} at $\partial\Omega$. Indeed, if $u_1 = u_2 \equiv 0$ in (3.1) (that correspond the case when $F_1 = F_2 \equiv 0$), then for component of the stress tensor on $\partial\Omega$ the equalities $\sigma_{3i} = \mu\partial u/\partial x_i$, $i = 1, 2$, $\sigma_{33} = 0$ take place. From these equalities follows that $\partial u/\partial n = \sigma_{3n}/\mu$ on $\partial\Omega$.

Here, we obtain a stability estimate for the inverse problem. It based on the method proposed in [16]. The essence of it is concluded in a construction of some amplitude relations on the characteristic surface $t = \tau(x)$, where $\tau(x)$ is the solution to the Cauchy problem

$$|\nabla \tau(x)|^2 = \frac{1}{\mu(x)}; \quad \tau|_{x_1=0} = 0, \tag{3.24}$$

and obtaining a priory estimates for the solution to (3.3) with data (3.23).

Define $S = S(T)$ as $S(T) = \{(x,t) \in (\partial\Omega \times \mathbb{R}) |\, \tau(x) \le t \le T + \tau(x)\}$, where T is an positive number. Assume, that the Riemannian metric $d\tau = \sqrt{(dx_1^2 + dx_2^2)/\mu(x)}$ has non-positive curvature in Ω. The sufficient condition of it is fulfilment of the following inequality (see section 4):

$$\Delta \ln \mu(x) \le 0, \quad x \in \Omega.$$

Let, moreover, Ω be convex with respect to geodesics of the metric. Then an arbitrary pair of points x and y can be joined in Ω by an unique geodesic.

For fixed numbers $q_0 > 0$ and $d > 0$, denote by $\Lambda(q_0, d)$ the set of functions (μ_0, p), satisfying the following conditions:

(1) $\mathrm{supp}(p_0(x), \mu(x) - 1) \subset \Omega$, $\inf_\Omega \tau(x) \ge d$,

(2) $\|p_0\|_{\mathbf{C}^{17}(\Omega)} \le q_0, \quad \|\mu - 1\|_{\mathbf{C}^{19}(\Omega)} \le q_0, \ \|k - 1\|_{\mathbf{C}^{11}[0,\infty)} \le q_0$.

For the solution to the direct problem (3.4), (3.4) the following lemma holds.

Lemma 3.1. *For arbitrary $T_0 > 0$ there exists $q_0(T_0) > 0$ such that for any $(\mu, p_0) \in \Lambda(q_0(T_0), d)$ solution to (3.3), (3.4) is represented in $K(T_0) = \{(x,t) |\, 0 < t \le T_0 - \tau(x)\}$ as*

$$u(x,t) = \alpha_0(x)\theta_0(t - \tau(x)) + \alpha_1(x)\theta_1(t - \tau(x)) + \alpha_2(x)\theta_2(t - \tau(x)) + v(x,t), \tag{3.25}$$

where $\theta_0(t)$ is the Heaviside function, while $\theta_1(t) = t\,\theta_0(t)$, $\theta_2(t) = t^2\,\theta_0(t)/2$. The coefficients $\alpha_0(x)$, $\alpha_1(x)$ and $\alpha_2(x)$ satisfy in $\{x \in \mathbb{R}^2 |\, x_1 \le d\}$ the conditions

$$\alpha_0(x) = \frac{1}{2}, \quad \alpha_1(x) = 0, \quad \alpha_2(x) = 0, \quad x_1 \le d. \tag{3.26}$$

Outside of this domain $\alpha_0(x)$ is defined by the formulae

$$\alpha_0(x) = \frac{1}{2}\exp(\varphi(x)), \quad \varphi(x) = \frac{1}{2}\int\limits_{\Gamma(x)} \Big[p_0(\xi) - \mathrm{div}\,(\mu(\xi)\nabla\tau(\xi))\Big]\, d\tau. \tag{3.27}$$

Here, $\Gamma(x)$ is the geodesic concluded on the plane ξ_1, ξ_2 between x and the line $\xi_1 = d$, which it intersects orthogonally, $d\tau$ is an element of the Riemannian length. The coefficients $\alpha_1(x)$ and $\alpha_2(x)$ are solutions to the following Cauchy problems:

$$\begin{aligned} 2\mu(x)\nabla\alpha_1(x)\cdot\nabla\tau(x) + \alpha_1(x)[\mathrm{div}(\mu(x)\nabla\tau(x)) - p_0(x)] \\ -\mathrm{div}[\mu(x)\nabla\alpha_0(x) - p_0(x)\mu(x)\alpha_0(x)\nabla\tau(x)] \\ +p_0(x)[\mu(x)\nabla\alpha_0(x)\cdot\nabla\tau(x) - k'(0)\alpha_0(x)] = 0, \quad \alpha_1|_{x_1=d} = 0, \end{aligned} \tag{3.28}$$

$$\begin{aligned} 2\mu(x)\nabla\alpha_2(x)\cdot\nabla\tau(x)+\alpha_2(x)[\mathrm{div}(\mu(x)\nabla\tau(x))-p_0(x)] \\ -\mathrm{div}\Big(\mu(x)\nabla\alpha_1(x)-p_0(x)\mu(x)[\alpha_1(x)\nabla\tau(x) \\ -\nabla\alpha_0(x)+k'(0)\alpha_0(x)\nabla\tau(x)]\Big) \\ +p_0(x)\Big[\mu(x)\nabla\alpha_1(x)\cdot\nabla\tau(x)+k'(0)\big(\mu(x)\nabla\alpha_0(x)\cdot\nabla\tau(x) \\ -\alpha_1(x)\big)-k''(0)\alpha_0(x)\Big]=0, \quad \alpha_2|_{x_1=d}=0. \end{aligned} \tag{3.29}$$

Functions $\alpha_k \in \mathbf{C}^{17-2k}(D(T_0))$, $k=0,1,2$, $D(T_0)=\{x|\, d\le\tau(x)\le T_0\}$, while $v(x,t)$ vanishes for $t<\tau(x)$. Functions $D_t^j v(x,t)$, $j=0,1,2$, $D_t^j=\partial^j/\partial t^j$ belong to $\mathbf{C}^{4-j}(\hat K(T_0))$, $\hat K(T_0)=\{(x,t)\in K(T_0)|\, t\ge\tau(x)\}$. Moreover, there exists a positive constant C such that for all $q_0\le q_0(T_0)$ the following inequalities hold:

$$\begin{aligned} \|\alpha_0(x)-1/2\|_{\mathbf{C}^{17}(D(T_0))} &\le Cq_0, \\ \|\alpha_k(x)\|_{\mathbf{C}^{17-2k}(D(T_0))} &\le Cq_0,\ k=1,2, \\ \|D_t^j\hat v(x,t)\|_{\mathbf{C}^{4-j}(K(T_0))} &\le Cq_0,\ j=0,1,2. \end{aligned}$$

The proof of this Lemma is given in [23]. The main result which proved therein is the following stability theorem.

Theorem 3.2. *Let $(\mu,p_{0k})\in\Lambda(q_0,d)$, $k=1,2$, while (g_k,h_k) be the Cauchy data corresponding solutions to (3.3), (3.4) for $p(x,t)=k(t)p_{0k}(x)$. Let, moreover, Ω is contained in some Riemannian circle of the radius ρ and the condition $T>4\rho$ holds. Then there are exist positive numbers q_0^* and C such that for $q_0\le q_0^*$ and arbitrary $(\mu_0,p_k)\in\Lambda(q_0,d)$, $k=1,2$, the following inequality valid*

$$\begin{aligned} \|p_{01}-p_{02}\|^2_{\mathbf{H}^1(\Omega)} \le C\Big[&\sum_{j=0}^{2}\Big(\|D_t^j(g_1-g_2)\|^2_{\mathbf{H}^1(S)}+\|D_t^j(h_1-h_2)\|^2_{\mathbf{L}^2(S)}\Big) \\ &+\sum_{j=0}^{1}\|D_t^j(g_1-g_2)\|^2_{\mathbf{L}^2(S_0)}\Big], \end{aligned}$$

in which $D_t^j=\partial^j/\partial t^j$, $S_0=\{(x,t)\in S|\, t=\tau(x)\}$.

As a corollary, we obtain the uniqueness theorem.

Theorem 3.3. *Let the conditions of the previous theorem be satisfied and $g_1=g_2$, $h_1=h_2$. Then there exists $q_0>0$ such that for any $(\mu,p_{01})\in\Lambda(q_0,d)$ and $(\mu,p_{02})\in\Lambda(q_0,d)$ the equality $p_{01}(x)=p_{02}(x)$ holds in Ω.*

3.2 3D inverse problem

In what follows, we will be interested in the problem of determining the functions $\lambda(x)$, $\mu(x)$, $p(x,t)$ and $q(x,t)$ from a given information about the family of solutions to some direct problems for (3.1). We will assume that $\rho(x)$ is a given function (for example, a given constant). The inverse problems of determining the kernels $p(x,t)$ and $q(x,t)$ in (3.1) under the assumption that $p(x,t)$ and $q(x,t)$ are representable in the forms $p(x,t) = k_1(t)p_0(x)$ and $q(x,t) = k_2(t)q_0(x)$, where $k_1(t)$ and $k_2(t)$ are given functions, while $p_0(x)$ and $q_0(x)$ are unknown functions whose support lies in some compact domain were studied earlier in [10, 21]. Here no special form of $p(x,t)$ and $q(x,t)$ is assumed. The constance of this subsection is based on the paper [26].

Formulate the posing of the inverse problem under consideration. Suppose that, in (3.1), the function F has the form

$$F(x,t;y) = f^0 \delta(x-y,t), \tag{3.30}$$

where $y \in \mathbb{R}^3$ is a point, a parameter of the problem, $\delta(x-y,t)$ is the Dirac delta-function, and $f^0 = (f_1^0, f_2^0, f_3^0)$ is the numerical vector characterizing the direction of the force concentrated at the point $(y,0)$. Let $u(x,t;y)$ be a solution to (3.1) satisfying the condition (3.2). Assume that, for some $\varepsilon \in (0,1)$, outside the domain $B_\varepsilon = \{x \in \mathbb{R}^3 |\, |x| < 1-\varepsilon\}$, the functions $\lambda(x)$, $\mu(x)$ and $\rho(x)$ coincide with given constants λ_0, μ_0 and ρ_0 respectively and $\lambda_0 + \mu_0 > 0$, $\mu_0 > 0$, $\rho_0 > 0$, and the functions $p(x,t)$ and $q(x,t)$ are identically zero for x outside B_ε for every $t \geq 0$.

Statement of the Inverse Problem. Suppose that, for some positive number $T > 0$, the function $(x,y,t) \in D(T)$, $D(T) = \{(x,y,t) |\, (x,y) \in (\partial B_0 \times \partial B_0), t \in [0, \tau_s(x,y) + T],\}$, $\partial B_0 = \{x \in \mathbb{R}^3 | |x| = 1\}$,

$$u(x,t;y) = f(x,y,t),\ (x,y,t) \in D(T). \tag{3.31}$$

From a given function $f(x,y,t)$, find $\lambda(x)$ and $\mu(x)$ in B_ε and $p(x,t)$ and $q(x,t)$ in $B_\varepsilon \times [0,T]$.

For a homogeneous medium, when ρ, λ, and μ are constant, $p = q = 0$, problem (3.1), (3.2) was solved by Love [11]. Its solution is given by the formula

$$\begin{aligned} u(x,t,y) &= \frac{f^0}{4\pi\rho\, c_s^2 |x-y|}\,\delta\big(t - \tau_s(x,y)\big) \\ &+ \frac{1}{4\pi\rho}\nabla \mathrm{div}\Big\{\frac{f^0}{|x-y|}\,\big[\theta_1\big(t-\tau_p(x,y)\big) - \theta_1\big(t-\tau_s(x,y)\big)\big]\Big\}, \end{aligned} \tag{3.32}$$

where $\theta_1(t) = t\theta_0(t)$ and $\theta_0(t)$ is the Heaviside function.

Introduce an infinite system of functions that were obtained from the Heaviside function by the consecutive integration or differentiation:

$$\theta_k(t) = \frac{t^k}{k!}\theta_0(t), \quad k = 1, 2, \ldots, \quad \theta_{-k}(t) = \frac{d^k}{dt^k}\theta_0(t) = \delta^{(k-1)}(t), \quad k = 1, 2, \ldots.$$

Note that the functions of this system satisfy the equality $\theta_k'(t) = \theta_{k-1}(t)$ for each $k = 0, \pm 1, \pm 2, \ldots$. For a homogeneous medium in which $\rho = \rho(y)$, $c_p = c_p(y)$ and $c_s = c_s(y)$, we can represent (3.32) as the finite ray expansion

$$\begin{aligned} u(x,t,y) = \sum_{k=-1}^{1} \Big[&\alpha^{(k,p)}(x,y)\,\theta_k(t-\tau_p(x,y)) \\ &+ \alpha^{(k,s)}(x,y)\,\theta_k(t-\tau_s(x,y))\Big], \end{aligned} \tag{3.33}$$

in which the coefficients $\alpha^{(k,p)}(x,y)$ are calculated by the formulas

$$\begin{aligned}
\alpha^{(-1,p)}(x,y) &= -\frac{(f^0\cdot\nabla_y\tau_p(x,y))\nabla\tau_p(x,y)}{4\pi\rho(y)\,c_p(y)\tau_p(x,y)},\\
\alpha^{(0,p)}(x,y) &= \frac{\nabla\tau_p(x,y)\times(f^0\times\nabla_y\tau_p(x,y)) - 2(f^0\cdot\nabla_y\tau_p(x,y))\nabla\tau_p(x,y)}{4\pi\rho(y)\,c_p(y)\tau_p^2(x,y)},\\
\alpha^{(1,p)}(x,y) &= \frac{\nabla\tau_p(x,y)\times(f^0\times\nabla_y\tau_p(x,y)) - 2(f^0\cdot\nabla_y\tau_p(x,y)))\nabla\tau_p(x,y)}{4\pi\rho(y)\,c_p(y)\tau_p^3(x,y)},
\end{aligned}\tag{3.34}$$

and the coefficients $\alpha^{(k,s)}(x,y)$ are computed by the formulas

$$\begin{aligned}
\alpha^{(-1,s)}(x,y) &= -\frac{\nabla\tau_s(x,y)\times(f^0\times\nabla_y\tau_s(x,y))}{4\pi\rho(y)\,c_s(y)\tau_s(x,y)},\\
\alpha^{(0,s)}(x,y) &= \frac{2(f^0\cdot\nabla_y\tau_s(x,y))\nabla\tau_s(x,y) - \nabla\tau_s(x,y)\times(f^0\times\nabla_y\tau_s(x,y))}{4\pi\rho(y)\,c_s(y)\tau_s^2(x,y)},\\
\alpha^{(1,s)}(x,y) &= \frac{2(f^0\cdot\nabla_y\tau_s(x,y))\nabla\tau_s(x,y) - \nabla\tau_s(x,y)\times(f^0\times\nabla_y\tau_s(x,y))}{4\pi\rho(y)\,c_s(y)\tau_s^3(x,y)}.
\end{aligned}\tag{3.35}$$

By the above assumption that the medium is homogeneous in some neighborhood of the source, the solution to (3.1), (3.2) coincides with its solution for a homogeneous medium in a sufficiently small neighborhood of $(y,0)$. In [27], we established an infinite asymptotic expansion of the solution to (3.1), (3.2) for an inhomogeneous medium similar to (3.33). The assertion is given as Lemma 3.2 below. This expansion is an "expansion with respect to smoothness" (the term is due to V. M. Babich [4]) in a neighborhood of the characteristic cones $t=\tau_p(x,y)$, $t=\tau_s(x,y)$ and is a basis for the study of the above-posed inverse problem.

We say that the set of functions $\rho(x)$, $\lambda(x)$, $\mu(x)$, $p(x,t)$ and $q(x,t)$ belongs to $\mathcal{P}$, $(\rho,\lambda,\mu,p,q)\in\mathcal{P}$, if the following hold:

(1) $\rho(x)$, $\mu(x)$ and $\lambda(x)+\mu(x)$ are positive functions for $x\in\mathbb{R}^3$ and coincide with positive constants ρ_0, μ_0, $\lambda_0+\mu_0$ outside the domain $B_\varepsilon=\{x\in\mathbb{R}^3\,|\,|x|<1-\varepsilon\}$, $\varepsilon\in(0,1)$;

(2) the functions $\rho(x)$, $\mu(x)$, $\lambda(x)$ and $p(x,t)$, $q(x,t)$ are infinitely differentiable with respect to their arguments for all x and t;

(3) for all $t\in[0,T)$, $T>0$, the supports of $p(x,t)$ and $q(x,t)$ are included in B_ε and $\operatorname{supp}(\rho(x)-\rho_0,\lambda(x)-\lambda_0,\mu(x)-\mu_0)\subset B_\varepsilon$;

(4) the metrics $d\tau_p=|dx|/c_p(x)$, $d\tau_s=|dx|/c_s(x)$ are simple in $x\in\mathbb{R}^3$.

Introduce some additional notation. Let $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ be a vector depending on the space variables x and y and let τ be a scalar function of these variables. Denote by $\kappa_{ij}^m(\alpha,\tau)$ the complex functions of x and y defined for $i,j=1,2,3$ and integers m

by the equalities

$$\begin{aligned}
\kappa_{ij}^0(\alpha,\tau) &= -\lambda(x)\delta_{ij}(\alpha\cdot\nabla\tau)-\mu(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j}+\alpha_j\frac{\partial\tau}{\partial x_i}\Big),\\
\kappa_{ij}^1(\alpha,\tau) &= \lambda(x)\delta_{ij}\mathrm{div}\alpha+\mu(x)\Big(\frac{\partial\alpha_i}{\partial x_j}+\frac{\partial\alpha_j}{\partial x_i}\Big)\\
&\quad -p_0(x)\delta_{ij}(\alpha\cdot\nabla\tau)-q_0(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j}+\alpha_j\frac{\partial\tau}{\partial x_i}\Big),\\
\kappa_{ij}^m(\alpha,\tau) &= p_{(m-2)}(x)\delta_{ij}\mathrm{div}\alpha+q_{(m-2)}(x)\Big(\frac{\partial\alpha_i}{\partial x_j}+\frac{\partial\alpha_j}{\partial x_i}\Big)\\
&\quad -p_{(m-1)}(x)\delta_{ij}(\alpha\cdot\nabla\tau)-q_{(m-1)}(x)\Big(\alpha_i\frac{\partial\tau}{\partial x_j}+\alpha_j\frac{\partial\tau}{\partial x_i}\Big),\ m\ge 2,
\end{aligned}\tag{3.36}$$

in which

$$p_m(x)=\frac{\partial^m p(x,t)}{\partial t^m}\Big|_{t=0},\quad q_m(x)=\frac{\partial^m q(x,t)}{\partial t^m}\Big|_{t=0}.$$

Let $Q^{(n,p)}$ and $Q^{(n,s)}$ be the vector functions whose components $Q_i^{(n,p)}$ and $Q_i^{(n,s)}$, $i=1,2,3$, for $n=1,2,\dots$ are calculated by the formulas

$$\begin{aligned}
Q_i^{(n,p)} &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1}\Big[\kappa_{ij}^m(\alpha^{(n-m,p)},\tau_p)\frac{\partial\tau_p}{\partial x_j}-\frac{\partial}{\partial x_j}\kappa_{ij}^{m-1}(\alpha^{(n-m,p)},\tau_p)\Big],\\
Q_i^{(n,s)} &= -\sum_{j=1}^{3}\sum_{m=2}^{n+1}\Big[\kappa_{ij}^m(\alpha^{(n-m,s)},\tau_s)\frac{\partial\tau_s}{\partial x_j}-\frac{\partial}{\partial x_j}\kappa_{ij}^{m-1}(\alpha^{(n-m,s)},\tau_s)\Big].
\end{aligned}\tag{3.37}$$

Assume that $Q^{(0,p)}=0$, $Q^{(0,s)}=0$. Denote by

$$\begin{aligned}
\zeta^p &= (\zeta_1^p,\zeta_2^p,\zeta_3^p)=-c_p^2(y)\tau_p(x,y)\nabla_y\tau_p(x,y),\\
\zeta^s &= (\zeta_1^s,\zeta_2^s,\zeta_3^s)=-c_s^2(y)\tau_s(x,y)\nabla_y\tau_s(x,y)
\end{aligned}$$

the Riemannian coordinates of a point x with respect to y in the metrics $d\tau_p=|dx|/c_p(x)$ and $d\tau_s=|dx|/c_s(x)$, respectively and designate as

$$J_p(x,y)=\det\left(\frac{\partial\zeta^p}{\partial x}\right),\quad J_s(x,y)=\det\left(\frac{\partial\zeta^s}{\partial x}\right)$$

the Jacobians of the transformations from the Riemannian coordinates to the Cartesian coordinates. Define a scalar function $A^{(p)}(x,y)$ and a matrix $T^{(s)}(x,y)$ by the equalities

$$A^{(p)}(x,y)=\frac{\sqrt{J_p(x,y)}}{4\pi\tau_p(x,y)c_p^2(y)\sqrt{\rho(x)\rho(y)}}\exp\left(\frac{1}{2}\int\limits_{\Gamma_p(x,y)}\frac{p_0(\xi)+2q_0(\xi)}{\lambda(\xi)+2\mu(\xi)}\,d\tau_p'\right),\tag{3.38}$$

$$T^{(s)}(x,y)=\frac{\mathcal{S}(x,y)\sqrt{J_s(x,y)}}{4\pi\tau_s(x,y)c_s^2(y)\sqrt{\rho(x)\rho(y)}}\exp\left(\frac{1}{2}\int\limits_{\Gamma_s(x,y)}\frac{q_0(\xi)}{\mu(\xi)}\,d\tau_s'\right).\tag{3.39}$$

In these equalities ξ is a variable point of the geodesics $\Gamma_p(x,y)$ and $\Gamma_s(x,y)$, respectively, $\tau_p' = \tau_p(\xi,y)$, $\tau_s' = \tau_s(\xi,y)$ and $\mathcal{S}(x,y)$ is the matrix exponent:

$$\mathcal{S}(x,y) = \exp\left\{ \int\limits_{\Gamma_s(x,y)} (\nabla \ln c_s(\xi))^t \, d\xi \right\},$$

in which $(\nabla \ln c_s(\xi))^t$ is a column vector and $d\xi = (d\xi_1, d\xi_2, d\xi_3)$ is a row vector, and their multiplication is carried out by the rules of matrix algebra.

Lemma 3.2. *Suppose that $(\rho,\lambda,\mu,p,q) \in \mathcal{P}$, $y \in \partial B_0$. Then the solution to (3.1), (3.2) is representable as the asymptotic series*

$$u(x,t,y) = \sum_{k=-1}^{\infty} \Big[\alpha^{(k,p)}(x,y)\,\theta_k(t-\tau_p(x,y)) + \alpha^{(k,s)}(x,y)\,\theta_k(t-\tau_s(x,y))\Big], \tag{3.40}$$

in which $\alpha^{(k,p)}(x,y)$ and $\alpha^{(k,s)}(x,y)$ are functions of class $\mathbf{C}^\infty(\mathbb{R}^6 \setminus \{(y,y)\})$ defined by (3.34), (3.35) for $|x-y| < \varepsilon$ and calculated for $|x-y| > \varepsilon$ by the formulas

$$\alpha^{(k,p)}(x,y) = c_p(x)[A^{(k,p)}(x,y)\nabla\tau_p(x,y) + \nabla\tau_p(x,y) \times B^{(k,p)}(x,y)],$$
$$\alpha^{(k,s)}(x,y) = c_s(x)[A^{(k,s)}(x,y)\nabla\tau_s(x,y) + \nabla\tau_s(x,y) \times B^{(k,s)}(x,y)],$$

in which

$$A^{(-1,p)}(x,y) = -(f^0 \cdot \nabla_y \tau_p(x,y))A^{(p)}(x,y), \quad B^{(-1,p)}(x,y) = 0, \tag{3.41}$$
$$B^{(-1,s)}(x,y) = -(f^0 \times \nabla_y \tau_s(x,y))T^{(s)}(x,y), \quad A^{(-1,s)}(x,y) = 0, \tag{3.42}$$

and the subsequent coefficients are computed by the recurrent formulas

$$A^{(n-1,p)}(x,y) = \left[\frac{A^{(n-1,p)}(\xi_p(x,y),y)}{A^{(p)}(\xi_p(x,y),y)} + \int\limits_{\Gamma_p(x,\xi_p(x,y))} \frac{R^{(n,p)}(\xi,y)}{2A^{(p)}(\xi,y)}\,d\tau_p'\right] A^{(p)}(x,y), \; n \geq 1,$$

$$B^{(n,p)}(x,y) = \frac{\lambda+2\mu}{\rho(\lambda+\mu)}\Big(c_p Q^{(n,p)} \times \nabla\tau_p - [\mu\Delta\tau_p + \nabla\mu\cdot\nabla\tau_p - q_0 c_p^{-2}]B^{(n-1,p)}$$
$$-c_p\Big[2\mu(\nabla\tau_p\cdot\nabla)\alpha^{(n-1,p)} + \nabla\big((\lambda+\mu)c_p^{-1}A^{(n-1,p)}\big) - c_p^{-1}A^{(n-1,p)}\nabla\mu\Big] \times \nabla\tau_p\Big), \; n \geq 0,$$

$$B^{(n-1,s)}(x,y) = \Bigg[B^{(n-1,s)}(\xi_s(x,y),y)T^{(s)}(\xi_s(x,y),y)$$
$$+\frac{1}{2}\int\limits_{\Gamma_s(x,\xi_s(x,y))} R^{(n,s)}(\xi,y)T^{(s)}(\xi,y)\,d\tau_s'\Bigg](T^{(s)})^{-1}(x,y), \; n \geq 1,$$

$$A^{(n,s)}(x,y) = \frac{c_s^2}{\lambda+\mu}\big\{[(\lambda+\mu)\mathrm{div}\alpha^{(n-1,s)} + \nabla\mu\cdot\alpha^{(n-1,s)} - (p_0+q_0)c_s^{-1}A^{(n-1,s)}]c_s^{-1}$$
$$+[\mu\Delta\tau_s + \nabla\mu\cdot\nabla\tau_s - q_0 c_s^{-2}]A^{(n-1,s)} + [2\mu c_s(\nabla\tau_s\cdot\nabla)\alpha^{(n-1,s)}$$
$$+c_s\nabla\big((\lambda+\mu)c_s^{-1}A^{(n-1,s)}\big) - A^{(n-1,s)}\nabla\mu - c_s Q^{(n,s)}]\cdot\nabla\tau_s\big\}, \; n \geq 0.$$

In these formulas $\xi_p(x,y)$ and $\xi_s(x,y)$ stand for the intersection points of the geodesics $\Gamma_p(x,y)$ and $\Gamma_s(x,y)$, respectively, with the sphere $|x-y|=\varepsilon$, the scalar function $R^{(n,p)}$ and the vector function $R^{(n,s)}$ are defined by the equations

$$\begin{aligned} R^{(n,p)} &= \frac{1}{\rho}\Big(c_p Q^{(n,p)}\cdot\nabla\tau_p - [(\lambda+\mu)c_p^{-1}\mathrm{div}(c_p\nabla\tau_p\times B^{(n-1,p)}) \\ &\quad +\nabla\mu\cdot(\nabla\tau_p\times B^{(n-1,p)}) + 2\mu(\nabla\tau_p\times B^{(n-1,p)})\cdot\nabla\ln c_p]\Big), \\ R^{(n,s)} &= \frac{1}{\rho}\{c_s Q^{(n,s)} + A^{(n-1,s)}[(\lambda+2\mu)\nabla\ln c_s - \nabla\lambda] - (\lambda+\mu)\nabla A^{(n-1,s)}\}\times\nabla\tau_s, \end{aligned}$$

moreover,

$$\frac{A^{(n-1,p)}(\xi_p(x,y),y)}{A^{(p)}(\xi_p(x,y),y)} = -\frac{2(f^0\cdot\nabla_y\tau_p(x,y))[c_p(y)]^n}{|\xi_p(x,y)-y|^n}\begin{cases}1, & n=1,2,\\ 0, & n>2,\end{cases}$$

$$B^{(n-1,s)}(\xi_s(x,y),y)T^{(s)}(\xi_s(x,y),y) = -\frac{(f^0\times\nabla_y\tau_s(x,y))[c_s(y)]^n}{|\xi_s(x,y)-y|^n}\begin{cases}1, & n=1,2,\\ 0, & n>2.\end{cases}$$

Use Lemma 3.2 for calculating the auxiliary series for the function $f(x,y,t)$ defined by (3.31). Let

$$\hat f(x,y,t) = \int_0^t f(x,y,z)\,dz, \quad t>0.$$

Given $x\in\partial B_0$, $y\in\partial B_0$, $x\neq y$, put

$$[\hat f]_{t=t_0}(x,y) = \lim_{t=t_0+0}\hat f(x,y,t) - \lim_{t=t_0-0}\hat f(x,y,t).$$

From (3.41), (3.42) it follows that $A^{(-1,p)}\neq 0$ if $f^0\cdot\nabla_y\tau_p(x,y)\neq 0$ and $B^{(-1,s)}\neq 0$ if $f^0\times\nabla_y\tau_s(x,y)\neq 0$. For every fixed $y\in\partial B_0$, the equality $f^0\cdot\nabla_y\tau_s(x,y)=0$ is possible only at those $x\in\partial B_0$ that correspond to the geodesics $\Gamma_p(x,y)$ starting from y in directions orthogonal to f^0. The set of these points x forms a curve $l(y)$ lying on ∂B_0. The equality $f^0\times\nabla_y\tau_s(x,y)=0$ is possible for fixed $y\in\partial B_0$ only at the only point$x\in\partial B_0$ at which the geodesic $\Gamma_s(x,y)$ collinear to the vector f^0 at y intersects ∂B_0. At all points where

$$f^0\cdot\nabla_y\tau_p(x,y)\neq 0, \quad f^0\times\nabla_y\tau_s(x,y)\neq 0,$$

we have the equalities

$$\begin{aligned} \tau_p(x,y) &= \inf\{t_0|\, t_0>0,\ [\hat f]_{t=t_0}(x,y)\neq 0\}, \\ \tau_s(x,y) &= \inf\{t_0|\, t_0>\tau_p(x,y),\ [\hat f]_{t=t_0}(x,y)\neq 0\}. \end{aligned}$$

By the smoothness of $\tau_p(x,y)$ and $\tau_p(x,y)$, they are defined by these equalities for all $(x,y)\in\partial B_0\times\partial B_0$. Moreover, for $m=0,1,2,\ldots$, the equalities hold:

$$\left[\frac{\partial^m\hat f(x,y,t)}{\partial t^m}\right]_{t=\tau_p(x,y)} = \alpha^{(m-1,p)}(x,y), \quad \left[\frac{\partial^m\hat f(x,y,t)}{\partial t^m}\right]_{t=\tau_s(x,y)} = \alpha^{(m-1,s)}(x,y).$$

Thus, we have

Lemma 3.3. *The data of the inverse problem determine uniquely for all* $(x, y) \in \partial B_0 \times \partial B_0$ *the functions* $\tau_p(x, y)$ *and* $\tau_p(x, y)$ *and the infinite chain of the coefficients* $\alpha^{(k,p)}(x, y)$ *and* $\alpha^{(k,s)}(x, y)$ *occurring in expansion (3.33).*

Note that the functions $\tau_p(x, y)$ and $\tau_p(x, y)$ are uniquely determined by the definition of the wave propagation speeds of the longitudinal and transverse waves respectively, i.e., the definition of $c_p(x)$ and $c_s(x)$, and the functions $\alpha^{(n,p)}(x, y)$ and $\alpha^{(n,s)}(x, y)$ are uniquely determined by the definition of $c_p(x)$ and $c_s(x)$ and the functions $p_k(x)$ and $q_k(x)$ for all $k \leq n + 1$. Therefore, instead of the initial inverse problem, we may consider the problem of constructing the functions $c_p(x)$, $c_s(x)$ and $p_k(x)$, $q_k(x)$, $k = 0, 1, 2, \ldots$, inside B_ε from the functions $\tau_p(x, y)$, $\tau_s(x, y)$ and $\alpha^{(k,p)}(x, y)$, $\alpha^{(k,s)}$, $k = 0, 1, 2, \ldots$, defined for $(x, y) \in \partial B_0 \times \partial B_0$. This new problem splits into a sequence of problems: first we can find $c_p(x)$ from the given function $\tau_p(x, y)$ and then find $c_s(x)$ from $\tau_s(x, y)$, and then recurrently find $p_n(x)$ and $q_n(x)$ by using the family $\alpha^{(k,p)}(x, y)$, $\alpha^{(k,s)}$ for $-1 \leq k \leq n - 1$.

The problem of determining $c_p(x)$ inside B_0 from a function $\tau_p(x, y)$ as well as $c_s(x)$ from $\tau_s(x, y)$ given for $(x, y) \in \partial B_0 \times \partial B_0$ was studied for $\mathbb{R}^3$ in [5, 6, 13]. The results obtained in these articles imply the uniqueness of the determination of $c_p(x)$ and $c_s(x)$ inside B_0 from given $\tau_p(x, y)$ and $\tau_s(x, y)$. The so-found functions $c_p(x)$ and $c_s(x)$ determine $\tau_p(x, y)$ and $\tau_s(x, y)$ and also $\Gamma_p(x, y)$, $J_p(x, y)$ and $\Gamma_s(x, y)$, $J_s(x, y)$, and the matrix exponent $\mathcal{S}(x, y)$ for every $(x, y) \in \mathbb{R}^6$. Furthermore, since the density $\rho(x)$ is assumed to be defined, $c_p(x)$ and $c_s(x)$ uniquely determine the elasticity moduli $\lambda(x)$ and $\mu(x)$.

Consider the problem of the determination of the functions $p_n(x)$ and $q_n(x)$ from the family of functions $\alpha^{(k,p)}(x, y)$ and $\alpha^{(k,s)}(x, y)$, $-1 \leq k \leq n-1$,, defined for $(x, y) \in \partial B_0 \times \partial B_0$. For $n = 0$, from the given functions $\alpha^{(-1,p)}(x, y)$ and $\alpha^{(-1,s)}(x, y)$ we compute the functions $A^{(-1,p)}(x, y) = c_p(x)\alpha^{(-1,p)}(x, y) \cdot \nabla\tau_p(x, y)$ and $B^{(-1,s)}(x, y) = c_s(x)\alpha^{(-1,s)}(x, y) \times \nabla\tau_s(x, y)$ at $(x, y) \in \partial B_0 \times \partial B_0$. Using (3.38), (3.41) and (3.39), (3.42), we come to the equalities

$$\int\limits_{\Gamma_p(x,y)} \frac{p_0(\xi) + 2q_0(\xi)}{\lambda(\xi) + 2\mu(\xi)}\, d\tau_p' = g_0(x, y), \quad (x, y) \in \partial B_0 \times \partial B_0, \tag{3.43}$$

$$\int\limits_{\Gamma_s(x,y)} \frac{q_0(\xi)}{\mu(\xi)}\, d\tau_s' = h_0(x, y), \quad (x, y) \in \partial B_0 \times \partial B_0, \tag{3.44}$$

in which $g_0(x, y)$ and $h_0(x, y)$ are the given functions defined by the formulas

$$g_0(x, y) = 2 \ln \left\{ \frac{4\pi |A^{(-1,p)}(x, y)| \tau_p(x, y) c_p^2(y) \sqrt{\rho(x)\rho(y)}}{|f^0 \cdot \nabla_y \tau_p(x, y)| \sqrt{J_p(x, y)}} \right\},$$

$$h_0(x, y) = 2 \ln \left\{ \frac{4\pi |B^{(-1,s)}(x, y)| \tau_s(x, y) c_s^2(y) \sqrt{\rho(x)\rho(y)}}{|(f^0 \times \nabla_y \tau_p(x, y))\mathcal{S}(x, y)| \sqrt{J_s(x, y)}} \right\}.$$

The problems of constructing the function under integrals in (3.43), (3.44) are problems of integral geometry on the families of geodesics, the questions of the uniqueness and stability of whose solutions were studied in [5, 6, 14] The results in these articles imply the unique determination of the functions $(p_0+2q_0)/(\lambda+2\mu)$ and $q_0/\mu$by the right-hand sides of (3.43), (3.44). Since the elasticity moduli $\lambda(x)$and $\mu(x)$ are already found, the functions under integrals determine the functions $p_0(x)$ and $q_0(x)$ uniquely.

It is proved by induction that, for any $n\geq 1$, the functions $p_n(x)$ and $q_n(x)$ are defined uniquely by the coefficients $\alpha^{(k,p)}(x,y)$, $\alpha^{(k,s)}(x,y)$ defined on $(\partial B_0\times\partial B_0)$ for all $-1\leq k\leq n-1$. Indeed, assume that the functions $p_k(x)$ and $q_k(x)$ are already known for all $k\leq n-1$ and on $(\partial B_0\times\partial B_0)$ there are defined $\alpha^{(k,p)}(x,y)$, $\alpha^{(k,s)}(x,y)$ for $-1\leq k\leq n-1$. Then the known functions $p_k(x)$ and $q_k(x)$ determine $\alpha^{(k,p)}(x,y)$, $\alpha^{(k,s)}(x,y)$, $-1\leq k\leq n-2$, for all $(x,y)\in\mathbb{R}^6$. Further, from given $\alpha^{(n-1,p)}(x,y)$ and $\alpha^{(n-1,s)}(x,y)$, at the points $(x,y)\in\partial B_0\times\partial B_0$, we compute the functions

$$A^{(n-1,p)}(x,y)=c_p(x)\alpha^{(n-1,p)}(x,y)\cdot\nabla\tau_p(x,y),$$
$$B^{(n-1,s)}(x,y)=c_s(x)\alpha^{(n-1,s)}(x,y)\times\nabla\tau_s(x,y).$$

On the other hand, Lemma 3.2 defines formulas for their calculation via the scalar function $R^{(n,p)}(x,y)$ and the vector function $R^{(n,s)}(x,y)$, in which $p_n(x)$ and $q_n(x)$ occur implicitly. Easy calculations show that these functions admit the representations

$$R^{(n,p)}(x,y)=-\frac{p_n(x)+2q_n(x)}{\lambda(x)+2\mu(x)}A^{(-1,p)}(x,y)+\bar{R}^{(n,p)}(x,y),$$
$$R^{(n,s)}(x,y)=-\frac{q_n(x)}{\mu(x)}B^{(-1,s)}(x,y)+\bar{R}^{(n,s)}(x,y),$$

where $\bar{R}^{(n,p)}(x,y)$, $\bar{R}^{(n,s)}(x,y)$ and $\bar{R}^{(n,s)}(x,y)$, $\bar{R}^{(n,s)}(x,y)$ depend only on $p_k(x)$ and $q_k(x)$ for $k\leq n-1$ and $\alpha^{(k,p)}(x,y)$, $\alpha^{(k,s)}(x,y)$, $(x,y)\in\mathbb{R}^6$, for $-1\leq k\leq n-2$, and hence are known. Therefore, we arrive at the inequalities

$$\int\limits_{\Gamma_p(x,y)}\frac{p_n(\xi)+2q_n(\xi)}{\lambda(\xi)+2\mu(\xi)}\,d\tau_p'=g_n(x,y),\quad (x,y)\in\partial B_0\times\partial B_0, \tag{3.45}$$

$$\int\limits_{\Gamma_s(x,y)}\frac{q_n(\xi)}{\mu(\xi)}\,d\tau_s'=h_n(x,y),\quad (x,y)\in\partial B_0\times\partial B_0, \tag{3.46}$$

in which $g_n(x,y)$ and $h_n(x,y)$ are the given functions defined by the formulas

$$g_n(x,y)=\frac{2}{|(f^0\cdot\nabla_y\tau_p(x,y))|}\Bigg[\frac{A^{(n-1,p)}(x,y)}{A^{(p)}(x,y)}-\frac{A^{(n-1,p)}(\xi_p(x,y),y)}{A^{(p)}(\xi_p(x,y),y)}$$
$$-\int\limits_{\Gamma_p(x,\xi_p(x,y))}\frac{\bar{R}^{(n,p)}(\xi,y)}{2A^{(p)}(\xi,y)}\,d\tau_p'\Bigg],$$

$$h_n(x,y) = \frac{2}{|(f^0 \times \nabla_y \tau_s(x,y))\mathcal{S}(x,y)|} \Big| B^{(n-1,s)}(x,y)T^{(s)}(x,y) - B^{(n-1,s)}(\xi_s(x,y),y)T^{(s)}(\xi_s(x,y),y) - \frac{1}{2} \int\limits_{\Gamma_s(x,\xi_s(x,y))} \bar{R}^{(n,s)}(\xi,y)T^{(s)}(\xi,y)\, d\tau_s' \Big|.$$

In deriving (3.45) and (3.46), we have involved the fact that $p_n(x) = q_n(x) = 0$ on the parts of the geodesics $\Gamma_p(x,y)$ and $\Gamma_s(x,y)$ that belong to $B_0 \setminus B_\varepsilon$. The appearing problems of constructing solutions to (3.45), (3.46) are quite similar to the problems of integral geometry for $p_0(x)$ and $q_0(x)$. This implies the uniqueness of their solution. Thus, we have

Theorem 3.4. *Let $(\rho, \lambda, \mu, p, q) \in \mathcal{P}$. Then (3.31) uniquely determine functions $c_p(x)$, $c_s(x)$ and $\partial^k p(x,t)/\partial t^k|_{t=0} \equiv p_k(x)$, $\partial^k q(x,t)/\partial t^k|_{t=0} \equiv q_k(x)$, $k = 0,1,2,\ldots$, in B_ε*

Theorem 3.4 implies as a corollary a uniqueness theorem for a solution to the inverse problem.

Theorem 3.5. *Suppose that $(\rho, \lambda, \mu, p, q) \in \mathcal{P}$ and $p(x,t)$ and $q(x,t)$ are analytic functions with respect to t for $t \in [0,T)$, $T > 0$. Then (3.31) uniquely determine the functions $c_p(x)$, $c_s(x)$ for $x \in B_\varepsilon$ and $p(x,t)$, $q(x,t)$ for $(x,t) \in (B_\varepsilon \times [0,T))$*

Note that when $p(x,t)$ and $q(x,t)$ are polynomials in t, for their construction, it is required to find only finitely many functions $p_k(x)$ and $q_k(x)$. For calculating these functions, it suffices to use the finite ray expansion of the solution to (3.1), (3.2). For this it suffices that the coefficients of (3.1) have finite smoothness.

4 Appendix. Sufficient conditions of non-positivity of a curvature for the conformal Riemannian metric

We derive here a formula for the sectional curvature of the conformal Riemannian metric and clear the question when this curvature is non-positive. Let $x \in \mathbb{R}^n$, $n \geq 2$, and

$$ds^2 = g_{ij}(x)dx^i dx^j. \tag{4.1}$$

Above and hereafter we use the Einstein summation convention. The Levi-Civita connections of the Riemannian space coordinated to metric (4.1) are determined by the following formula (see, for instance, formula (94.9) in [1])

$$\Gamma^p_{ij} = \frac{1}{2} g^{lp} \Big(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \Big), \tag{4.2}$$

where (g^{ij}) is the inverse matrix to (g_{ij}). Components R_{lkij} of the curvature tensor are calculated then as follows (formula (110.4) in [1])

$$R_{lkij} = \frac{1}{2} \Big(\frac{\partial^2 g_{lj}}{\partial x_k \partial x_l} - \frac{\partial^2 g_{li}}{\partial x_k \partial x_j} - \frac{\partial^2 g_{kj}}{\partial x_l \partial x_i} + \frac{\partial^2 g_{ki}}{\partial x_l \partial x_j} \Big) + g_{pq}(\Gamma^p_{lj}\Gamma^q_{ki} - \Gamma^p_{kj}\Gamma^q_{li}). \tag{4.3}$$

Let a two-dimensional plane σ is given by the two orthogonal unit vectors $\nu = (\nu_1, \dots, \nu_n)$ and $\eta = (\eta_1, \dots, \eta_n)$,

$$g_{ij}(x)\nu^i\nu^j = 1, \quad g_{ij}(x)\eta^i\eta^j = 1, \qquad g_{ij}(x)\nu^i\eta^j = 0. \tag{4.4}$$

Then the sectional curvature $K(x, \sigma)$ at a point x and in the given two-dimensional direction σ is determined by the formula

$$K(x, \sigma) = R_{lkij}\nu^i\nu^l\eta^k\eta^j. \tag{4.5}$$

Use formulae (4.1)-(4.5) for a calculation of the sectional curvature $K(x, \sigma)$ for the case of the conformal Riemannian metric $ds^2 = g(x)|dx|^2$. We assume that g is a positive and twice continuously differentiable function in a domain Ω. In this case $g_{ij} = g(x)\delta_{ij}$, $g^{ij} = \delta_{ij}/g(x)$, and

$$\Gamma^p_{ij} = \frac{1}{2}\Big(\delta_{ip}\frac{\partial \ln g}{\partial x_j} + \delta_{jp}\frac{\partial \ln g}{\partial x_i} - \delta_{ij}\frac{\partial \ln g}{\partial x_p}\Big). \tag{4.6}$$

Then relations (4.4) are $g|\nu|^2 = 1$, $g|\eta|^2 = 1$, $\nu \cdot \eta = 0$. Here $\nu \cdot \eta$ means the scalar product of vectors ν and η. Taking these relations into account we find

$$\begin{aligned}
&\Big(\frac{\partial^2 g_{lj}}{\partial x_k \partial x_l} - \frac{\partial^2 g_{li}}{\partial x_k \partial x_j} - \frac{\partial^2 g_{kj}}{\partial x_l \partial x_i} + \frac{\partial^2 g_{ki}}{\partial x_l \partial x_j}\Big)\nu^i\nu^l\eta^k\eta^j \\
&= \Big(\delta_{lj}\frac{\partial^2 g}{\partial x_k \partial x_l} - \delta_{li}\frac{\partial^2 g}{\partial x_k \partial x_j} - \delta_{kj}\frac{\partial^2 g}{\partial x_l \partial x_i} + \delta_{ki}\frac{\partial^2 g}{\partial x_l \partial x_j}\Big)\nu^i\nu^l\eta^k\eta^j \\
&= -\frac{1}{g}\frac{\partial^2 g}{\partial x_i \partial x_j}(\nu^i\nu^j + \eta^i\eta^j) = -\Big(\frac{\partial^2 \ln g}{\partial x_i \partial x_j} - \frac{\partial \ln g}{\partial x_i}\frac{\partial \ln g}{\partial x_j}\Big)(\nu^i\nu^j + \eta^i\eta^j) \\
&= -\frac{\partial^2 \ln g}{\partial x_i \partial x_j}(\nu^i\nu^j + \eta^i\eta^j) + (\nabla \ln g \cdot \nu)^2 + (\nabla \ln g \cdot \eta)^2 \\
&= -\frac{\partial^2 \ln g}{\partial x_i \partial x_j}(\nu^i\nu^j + \eta^i\eta^j) + \frac{1}{g}|\nabla \ln g|^2.
\end{aligned} \tag{4.7}$$

To the other hand,

$$\begin{aligned}
\Gamma^p_{lj}\Gamma^q_{ki}\nu^i\nu^l\eta^k\eta^j \;&=\; \frac{1}{4}\Big(\delta_{lp}\frac{\partial \ln g}{\partial x_j} + \delta_{jp}\frac{\partial \ln g}{\partial x_l} - \delta_{lj}\frac{\partial \ln g}{\partial x_p}\Big)\nu^l\eta^j \\
&\quad \times\Big(\delta_{kq}\frac{\partial \ln g}{\partial x_i} + \delta_{iq}\frac{\partial \ln g}{\partial x_k} - \delta_{ki}\frac{\partial \ln g}{\partial x_q}\Big)\nu^i\eta^k \\
&=\; \frac{1}{4}\Big((\nabla \ln g \cdot \eta)\nu^p + (\nabla \ln g \cdot \nu)\eta^p\Big) \\
&\quad \times\Big((\nabla \ln g \cdot \nu)\eta^q + (\nabla \ln g \cdot \eta)\nu^q\Big),
\end{aligned} \tag{4.8}$$

$$\Gamma^p_{kj}\Gamma^q_{li}\nu^i\nu^l\eta^k\eta^j = \frac{1}{4}\Big(\delta_{kp}\frac{\partial \ln g}{\partial x_j} + \delta_{jp}\frac{\partial \ln g}{\partial x_k} - \delta_{kj}\frac{\partial \ln g}{\partial x_p}\Big)\eta^k\eta^j$$
$$\times\Big(\delta_{lq}\frac{\partial \ln g}{\partial x_i} + \delta_{iq}\frac{\partial \ln g}{\partial x_l} - \delta_{li}\frac{\partial \ln g}{\partial x_q}\Big)\nu^i\nu^l$$
$$= \frac{1}{4}\Big(2(\nabla \ln g\cdot\eta)\eta^p - \frac{1}{g}\frac{\partial \ln g}{\partial x_p}\Big)$$
$$\times\Big(2(\nabla \ln g\cdot\nu)\nu^q - \frac{1}{g}\frac{\partial \ln g}{\partial x_q}\Big). \tag{4.9}$$

From the latter formulae we obtain

$$g_{pq}(\Gamma^p_{lj}\Gamma^q_{ki} - \Gamma^p_{kj}\Gamma^q_{li})\nu^i\nu^l\eta^k\eta^j = \delta_{pq}\frac{g}{4}\Big[\Big((\nabla \ln g\cdot\eta)\nu^p + (\nabla \ln g\cdot\nu)\eta^p\Big)$$
$$\times\Big((\nabla \ln g\cdot\nu)\eta^q + (\nabla \ln g\cdot\eta)\nu^q\Big)$$
$$-\Big(2(\nabla \ln g\cdot\eta)\eta^p - \frac{1}{g}\frac{\partial \ln g}{\partial x_p}\Big)\Big(2(\nabla \ln g\cdot\nu)\nu^q - \frac{1}{g}\frac{\partial \ln g}{\partial x_q}\Big)\Big]$$
$$= \frac{1}{4}\Big[3(\nabla \ln g\cdot\nu)^2 + 3(\nabla \ln g\cdot\eta)^2 - \frac{1}{g}|\nabla \ln g|^2\Big] = \frac{1}{2g}|\nabla \ln g|^2. \tag{4.10}$$

Formulae (4.3), (4.5), (4.7), (4.10) imply that

$$K(x,\sigma) = -\frac{1}{2}\frac{\partial^2 \ln g}{\partial x_i \partial x_j}(\nu^i\nu^j + \eta^i\eta^j). \tag{4.11}$$

If $n = 2$ the latter formula can be written in a more simple form. Using that in this case $\eta_1 = -\nu_2$, $\eta_2 = \nu_1$, we find

$$K(x) = -\frac{1}{2g}\Delta \ln g. \tag{4.12}$$

Note that in the two-dimensional space there exists only the unique plane σ, that coincides with this space. Therefore the curvature does not depend on σ. Formula (4.12) coincides with the formula for the Gauss curvature of a surface in $\mathbb{R}^3$ equipped by isotropic metric $ds^2 = g(x)|dx|^2$ (see chapter 2, §13, Theorem 2 in [2]). The latter is completely agrees with the theory of the curvature for the two-dimensional Riemannian manifolds.

In the general case, it is easy to prove using (4.11), (4.12) that the sufficient condition for a non-positivity of $K(x,\sigma)$ can be given as

$$\begin{aligned} &\Delta \ln g \ge 0, && n = 2, \\ &\frac{\partial^2 \ln g}{\partial x_i \partial x_j}\nu^i\nu^j \ge 0, && n \ge 3. \end{aligned} \tag{4.13}$$

The manifold (Ω, g) is called by the manifold of a non-positivity curvature if $K(x,\sigma) \le 0$ for all $x \in \Omega$ and any two-dimensional planes σ.

It is well known [3] (Hadamard-Cartan theorem) that in any simply connected complete manifold of a nonpositivity curvature each two points can be joined by a single geodesic line. The latter is also true for compact manifolds with strongly convex (with respect to geodesics) borders. Thus, if conditions (4.13) is fulfilled for all $x \in \Omega$ and the boundary $\partial\Omega$ of the domain Ω is convex then any two points of $\overline{\Omega}$ are joined by a unique geodesic.

Related to this, we derive a sufficient condition for the strong convexity of boundary $\partial\Omega$ with respect to geodesics. Let the boundary given by the equation $F(x) = 0$, where F is a twice continuously differentiable function, and $F(x) < 0$ in Ω. Take an arbitrary point $x^0 \in \partial\Omega$ and consider a geodesic line passing throw point x^0 in a tangent direction ν, $\nu = (\nu^1, \dots, \nu^n)$, $g_{ij}(x^0)\nu^i\nu^j = 1$, to $\partial\Omega$. Let s be the length of the geodesic and $s = 0$ at x^0. Then an equation of the geodesic line can be presented in the parametric form as $x = x(s) = (x_1(s), \dots, x_n(s))$, where the function $x(s)$ solves the Cauchy problem

$$\ddot{x}_k = -\Gamma^k_{ij}\dot{x}^i\dot{x}^j, \quad k = 1, 2, \dots, n, \quad x(0) = x^0, \quad \dot{x}(0) = \nu. \tag{4.14}$$

Then the condition of the strong convexity of boundary $\partial\Omega$ at x^0 can be written as $F(x(s)) > 0$ for all sufficiently small $|s| > 0$. The latter is equivalent to the requirement

$$F(x^0) + F_{x_k}(x^0)\delta x^k + \frac{1}{2}F_{x_i x_j}(x^0)\delta x^i \delta x^j + o(|\delta x|^2) > 0, \quad as\ |\delta x| \to 0. \tag{4.15}$$

Here $\delta x = (\delta x^1, \dots, \delta x^n) = x(s) - x^0$. Taking into account that $F(x^0) = 0$, $\delta x^k = s\nu^k - s^2\Gamma^k_{ij}(x^0)\nu^i\nu^j/2 + o(s^2)$ and $F_{x_k}(x^0)\nu^k = 0$, we find that condition (4.15) is satisfied if

$$-F_{x_k}(x^0)\Gamma^k_{ij}(x^0)\nu^i\nu^j + F_{x_i x_j}(x^0)\nu^i\nu^j > 0, \quad \forall \nu \in \{\nu |\, g_{ij}(x^0)\nu^i\nu^j = 1,\ \nu \cdot \nabla F(x^0)\} \tag{4.16}$$

Hence, the border $\partial\Omega$ is strongly convex with respect to geodesics if the latter condition holds for all $x^0 \in \partial\Omega$.

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