EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 14, Issue 2 (2026) 35 – 51

SYNTHESIS OF POWER AND MOTION CONTROL ALONG GIVEN TRAJECTORIES OF PLATE HEATING SOURCES WITH OPTIMIZATION OF MEASUREMENT POINT PLACEMENT

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Abstract The problem of optimal synthesizing the power control of point heat sources moving along given trajectories for heating a two-dimensional plate is investigated. The motion of sources is described by ordinary differential equations. The current values of the power and movement velocity of the sources are determined based on the plate temperature values measured by the sensing devices. The source power, their movement velocity, and the locations of the measurement points are optimized. Necessary optimality conditions are derived for the feedback parameters that define the current values of the source power and movement velocity based on the measured temperature values on the plate and the positions of the measuring sensors. These conditions include formulas for the components of the gradient of the objective functional with respect to the optimized parameters. Results from computer experiments obtained using first-order numerical optimization methods are presented.

Keywords: plate heating, moving point source, given trajectory, feedback control, measuring sensor, feedback parameters.

AMS Mathematics Subject Classification: 49K15, 49K20.

DOI: 10.32523/2306-6172-2026-14-2-35-51

1 Introduction

The work focuses on the feedback control problem for the power and motion of point heat sources moving along prescribed trajectories to heat a plate. The current control values depend on the plate temperature values measured by sensors, whose placement locations also need to be optimized. The velocities of the point sources are controlled and are determined based on the currently measured temperature values.

Many works [1]–[5] are devoted to various aspects of the study of optimal control problems for objects with distributed parameters, described by boundary value problems with respect to partial differential equations. A special case of these problems are the problems of controlling lumped sources [1], [2], [6]–[8], which are frequently encountered in practice, among which the most important for practice are control problems with feedback. In contrast to the problems of feedback control of objects described by systems of ordinary differential equations [9]–[11], for objects with distributed parameters described by partial differential equations, both the theoretical aspects and practical methods for solving them have been significantly understudied on [3]–[5]. The difficulty of obtaining constructive results and their application in practice is associated with both theoretical problems of their research and technical problems of organizing feedback, operational processing of information, making optimal decisions and their technical feasibility.

The specificity of the problem of controlling the heating process of a plate using moving sources along given trajectories considered in this work is as follows. Firstly, the velocity of movement and the power of the sources at each moment in time are determined based on

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the results of temperature measurements at the measurement points. Secondly, and most importantly, the coordinates of the locations of the sensors taking measurements are optimized. Formulas are proposed that determine the linear dependencies of control actions on the movement and power of sources on the measured temperature values at the measurement points. With respect to the constant parameters (coefficients) involved in these dependencies and the coordinates of the measurement points, formulas for the corresponding components of the objective functional gradient were obtained. They allow us to formulate the necessary first-order optimality conditions for the parameters being optimized, and to use effective first-order numerical optimization methods to determine their values.

The results of numerous numerical experiments on the application of the proposed approach to solving illustrative problems are presented, and the analysis of the obtained results is carried out.

2 Problem statement

The problem of controlling the heating process of a thin two-dimensional plate, described by the parabolic differential equation, is considered [12]:

$$\frac{\partial u(x,t)}{\partial t} = a^2 \mathrm{div}\,(\mathrm{grad}\; u(x,t)) - \lambda_0 \left[u(x,t) - \theta\right] + \sum_{i=1}^{N_c} q_i(t)\,\delta\left(x - \eta^i(t)\right), \tag{1}$$

$$x \in \Omega, \quad t \in (t_0, t_f],$$

with boundary condition

$$\frac{\partial u(x,t)}{\partial \mathrm{n}} = \lambda \left[u(x,t) - \theta\right], \quad x \in \Gamma, \quad t \in (t_0, t_f]. \tag{2}$$

Here $u(x,t)$ is the temperature of the plate at the point $x \in \Omega$ at the time $t \in [t_0, t_f]$; Ω is the domain occupied by the plate with the boundary Γ; n is the internal normal to Γ; $a^2 > 0$, $\lambda_0 \geq 0$, $\lambda \geq 0$ are the given parameters of the heating process; θ is the ambient temperature; $\eta = \eta(t) = \left(\eta^1(t), \eta^2(t), \ldots, \eta^{N_c}(t)\right)$, $\eta^i(t) \in \Omega$, $i = 1, 2, \ldots, N_c$ are the coordinates of the moving lumped heat sources on the plate at the time $t \in [t_0, t_f]$; $\delta(\cdot)$ is a two-dimensional Dirac function such that for an arbitrary continuous function $f(x)$ in Ω and $\tilde{x} \in \Omega$ the following holds [1], [12]–[14]:

$$\iint_{\Omega} f(x)\,\delta(x - \tilde{x})\,dx = f(\tilde{x}).$$

The initial temperature distribution of the plate at time t_0 is uniform across all points, but is not precisely known, and is assumed to belong to a given set $\mathcal{B} \subset \mathbb{R}$:

$$u(x, t_0) = b(x) = b \in \mathcal{B}, \quad x \in \Omega. \tag{3}$$

In this case, the distribution density function $\rho_{\mathcal{B}}(b)$ is known such that

$$\rho_{\mathcal{B}}(b) \geq 0, \quad b \in \mathcal{B}, \quad \int_{\mathcal{B}} \rho_{\mathcal{B}}(b)\,db = 1.$$

The constant temperature of the external ambient θ, the exact value of which is not specified, but is known the set of their possible values Θ and the distribution density function $\rho_\Theta(\theta)$, is such that

$$\rho_\Theta(\theta) \geq 0, \quad \theta \in \Theta, \quad \int\limits_\Theta \rho_\Theta(\theta)\, d\theta = 1.$$

The sets $\mathcal{B}$ and Θ can be discrete:

$$\mathcal{B} = \{b_i: \quad i = 1, 2 \ldots, N_\mathcal{B}\}, \quad \Theta = \{\theta_j: \quad j = 1, 2 \ldots, N_\Theta\}, \tag{4}$$

with given probability values

$$p_i^b = P(b = b_i), \quad i = 1, 2, \ldots, N_\mathcal{B}, \quad p_j^\theta = P(\theta = \theta_j), \quad j = 1, 2, \ldots, N_\Theta.$$

$$\sum_{i=1}^{N_\mathcal{B}} p_i^b = 1, \quad \sum_{j=1}^{N_\Theta} p_j^\theta = 1.$$

The function $u(x,t)$, which determines the value of the plate temperature at the point $x \in \Omega$ at the time t, is a generalized unique solution of the problem (1)–(3), see [14], namely: for an arbitrary function $\psi(x,t) \in H^{1,1}(\Omega \times [t_0, t_f])$ such that

$$\psi(x, t_f) = 0, \quad x \in \Omega, \quad \frac{\partial \psi(x,t)}{\partial \mathrm{n}} = \lambda \psi(x,t), \quad x \in \Gamma, \quad t \in [t_0, t_f],$$

equality holds

$$\int\limits_{t_0}^{t_f} \iint\limits_\Omega u(x,t) \left[-\frac{\partial \psi(x,t)}{\partial t} - a^2 \mathrm{div}\left(\mathrm{grad}\, \psi(x,t)\right) + \lambda_0 \psi(x,t) \right] dxdt - \int\limits_{t_0}^{t_f} \iint\limits_\Omega \lambda_0 \theta \psi(x,t)\, dxdt$$

$$= \sum_{i=1}^{N_c} \int\limits_{t_0}^{t_f} \psi\left(\eta^i(t), t\right) q_i(t) dt + \iint\limits_\Omega \psi(x, t_0)\, b(x)\, dx - \int\limits_{t_0}^{t_f} \int\limits_\Gamma a^2 \lambda \theta \psi(x,t)\, dxdt.$$

The plate is heated by N_c moving point sources with controlled powers $q_i(t)$, $i = 1, 2, \ldots, N_c$, which are piecewise continuous functions of t satisfying the following conditions:

$$q(t) \in Q = \left\{q_i: \quad \underline{q_i} \leq q_i \leq \overline{q_i}, \quad i = 1, 2, \ldots, N_c\right\}, \quad t \in [t_0, t_f], \tag{5}$$

$$q(t) = (q_1(t), q_2(t), \ldots, q_{N_c}(t)),$$

which $\underline{q_i}$, $\overline{q_i}$, $i = 1, 2, \ldots, N_c$ are given.

We will assume that all point sources $\eta^i = \eta^i(t) = (\eta_1^i(t), \eta_2^i(t)) \in \Omega$, $i = 1, 2, \ldots, N_c$, move along the plate with individually defined and neither self-intersecting nor mutually intersecting trajectories.

$$S_i = \left\{\eta^i(t) \in \Omega: \ t \in [t_0, t_f]\right\}, \quad i = 1, 2, \ldots, N_c.$$

The trajectories are defined by the given functions $r_i = f_i(\varphi)$, where $\varphi \in [0, 2\pi]$, meaning that at each time instant, the state of the i-th source satisfies the relation below:

$$r_i(t) = f_i(\varphi_i(t)), \quad i = 1, 2, \ldots, N_c, \quad t \in [t_0, t_f]. \tag{6}$$

We will describe the movement of sources along given trajectories using polar coordinate systems local to each source. We define the origins of the coordinates of the polar systems for each source by the given points $\hat{\eta}^i = \left(\hat{\eta}_1^i, \hat{\eta}_2^i\right)$, $i = 1, 2, \ldots, N_c$, and we direct the directions of the axes of the polar angles parallel to the Ox axis of the main coordinate system. Then the positions of the sources in the original cartesian system on the plane at each moment of time $t \in [t_0, t_f]$ are determined as follows:

$$\eta_1^i(t) = \hat{\eta}_1^i + r_i(t)\cos(\varphi_i(t)), \quad \eta_2^i(t) = \hat{\eta}_2^i + r_i(t)\sin(\varphi_i(t)), \tag{7}$$

$$t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c.$$

The velocities of the sources along the prescribed trajectories serve as control values, that is, the angular velocity $\omega_i = \omega_i(t)$ is taken as the control variable for each i-th source:

$$\frac{\mathrm{d}\varphi_i(t)}{\mathrm{d}t} = \omega_i(t), \quad t \in (t_0, t_f], \quad i = 1, 2, \ldots, N_c. \tag{8}$$

$$\varphi_i(t_0) = \varphi_i^0, \quad i = 1, 2, \ldots, N_c. \tag{9}$$

φ_i^0, $i = 1, 2, \ldots, N_c$ are the given polar angle for the initial position of the i-th source.

Let a piecewise continuous vector function $\omega = \omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_{N_c}(t))$ satisfy the condition:

$$\omega(t) \in W = \left\{\omega_i : \quad \underline{\omega_i} \leq \omega_i \leq \overline{\omega_i}, \quad i = 1, 2, \ldots, N_c\right\}, \quad t \in [t_0, t_f], \tag{10}$$

where $\underline{\omega_i}$, $\overline{\omega_i}$, $i = 1, 2, \ldots, N_c$ are given.

The optimal control problem for the plate heating process consists of determining admissible control actions $\omega(t)$ governing the motion of the sources and their power $q(t)$, which minimize the following objective functional:

$$J(q, \omega) = \int\limits_{\Theta}\int\limits_{\mathcal{B}} I(q, \omega; b, \theta)\rho_{\mathcal{B}}(b)\,\rho_{\Theta}(\theta)\,dbd\theta, \tag{11}$$

$$I(q, \omega; b, \theta) = \iint\limits_{\Omega} \mu(x)\left[u(x, t_f) - U(x)\right]^2 dx \tag{12}$$

$$+\varepsilon_1\|q(t) - \hat{q}\|^2_{L_2^{N_c}[t_0,t_f]} + \varepsilon_2\|\omega(t) - \hat{\omega}\|^2_{L_2^{N_c}[t_0,t_f]}.$$

Here $u(x, t) = u(x, t; q, \omega, b, \theta)$ is the solution of the initial–boundary value problem (1)–(3) with the initial condition $u(x, t_0) = b$, the ambient temperature θ for admissible values of the source powers $q = q(t)$ and the impacts on their movements $\omega = \omega(t)$; $U(x)$ is the given temperature distribution of the plate that must be achieved at the end of the heating process; $\mu(x) \geq 0$, $x \in \Omega$, is the given weight function; ε_1, ε_2, $\hat{q}$, $\hat{\vartheta}$ are the given regularization parameters of the functional of the problem [14].

If the sets $\mathcal{B}$ and Θ are discrete sets (4), then the functional $J(q, \omega)$ from (11) takes the form:

$$J(q, \omega) = \sum_{j=1}^{N_\Theta}\sum_{i=1}^{N_{\mathcal{B}}} I(q, \omega; b_i, \theta_j)\,p_i^b p_j^\theta. \tag{13}$$

Similarly, the objective functional can be written in the case where one of the sets $\mathcal{B}$ or Θ is continuous and the other is finite.

The objective functionals (11), (12) and (13) measure, on average, the quality of the optimized parameters q and ω over all admissible initial plate temperatures $b \in \mathcal{B}$ and ambient temperature values $\theta \in \Theta$. In this case, the values of the distribution density functions $\rho_{\mathcal{B}}(b)$ and $\rho_\Theta(\theta)$ on the sets $\mathcal{B}$ and Θ are taken into account.

Let the current temperature values be measured continuously by sensors at the points of the plate $\xi^j = (\xi_1^j, \xi_2^j) \in \Omega$, $j = 1, 2, \ldots, N_o$ during the entire heating process:

$$\hat{u}_j(t) = u\left(\xi^j, t\right), \quad t \in [t_0, t_f], \quad \xi^j \in \Omega, \quad j = 1, 2, \ldots, N_o. \tag{14}$$

Let there be restrictions on the places of their installation due to technological and technical conditions:

$$\xi^j \in \Omega_j \subset \Omega, \quad j = 1, 2, \ldots, N_o, \tag{15}$$

where closed subsets $\Omega_j \subset \Omega$, $j = 1, 2, \ldots, N_o$ are given, and:

$$\Omega_j \cap S_i = \emptyset, \quad j = 1, 2, \ldots, N_o, \quad i = 1, 2, \ldots, N_c.$$

Note that it is possible to consider the case of taking measurements at given discrete moments of time, $\tau_s \in [t_0, t_f)$, $s = 0, 1, \ldots, N_t$, $\tau_0 = t_0$.

$$\check{u}_s^j = u\left(\xi^j, \tau_s\right), \quad j = 1, 2, \ldots, N_o, \quad s = 0, 1, \ldots, N_t. \tag{16}$$

It is proposed to use measurement value to determine the current control actions, namely the power of each source $q_i(t)$, $i = 1, 2, \ldots, N_c$, and the angular velocities of the moving point heat sources $\omega_i(t)$, $i = 1, 2, \ldots, N_c$.

To this end, we consider a linear feedback law for the control actions based on continuously measured temperature values $u\left(\xi^j, t\right)$.

$$q_i(t) = \sum_{j=1}^{N_o} k_{1,i}^j \left[u\left(\xi^j, t\right) - z_{1,i}^j\right], \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c, \tag{17}$$

$$\omega_i(t) = \sum_{j=1}^{N_o} k_{2,i}^j \left[u\left(\xi^j, t\right) - z_{2,i}^j\right], \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c, \tag{18}$$

in case of continuous measurements (14). In case of measurements at discrete moments in time (16), we use linear feedback of the form:

$$q_i(t) = \sum_{j=1}^{N_o} k_{1,i}^j \left[\check{u}_s^j - z_{1,i}^j\right], \quad \omega_i(t) = \sum_{j=1}^{N_o} k_{2,i}^j \left[\check{u}_s^j - z_{2,i}^j\right], \tag{19}$$

$$t \in [\tau_s, \tau_{s+1}), \quad s = 0, 1, \ldots, N_t, \quad \tau_{N_t+1} = t_f.$$

Here, the constants $k_{1,i}^j$, $z_{1,i}^j$, $k_{2,i}^j$, $z_{2,i}^j$, $i = 1, 2, \ldots, N_c$, $j = 1, 2, \ldots, N_o$ are treated as synthesized parameters. The feedback parameters $k_{1,i}^j$ and $k_{2,i}^j$, $i = 1, 2, \ldots, N_c$, $j = 1, 2, \ldots, N_o$, by analogy with synthesis problems for lumped-parameter systems, are referred to as gain coefficients, while the unknown nominal parameters $z_{1,i}^j$ and $z_{2,i}^j$, $i = 1, 2, \ldots, N_c$, $j = 1, 2, \ldots, N_o$ are primarily determined by the known function $U(x)$, $x \in \Omega$.

From the properties of differential equations (1) and (8), due to the continuity of the function $u(x, t)$ with respect to t for $x \in \Omega$, it follows that the control actions $(q(t), \omega(t))$, determined from (17), (18), are themselves continuous functions of time.

By substituting the control expressions (17) and (18), defined through the continuous feedback rule (14), into the differential equations (1) and (8), respectively, we obtain the following system of differential equations:

$$\frac{\partial u(x,t)}{\partial t} = a^2 \mathrm{div}\left(\mathrm{grad}\, u(x,t)\right) - \lambda_0 \left[u(x,t) - \theta\right] \tag{20}$$

$$+\sum_{i=1}^{N_c} \delta\left(x - \eta^i(t)\right) \sum_{j=1}^{N_o} k_{1,i}^j \left[u\left(\xi^j, t\right) - z_{1,i}^j\right], \quad x \in \Omega, \quad t \in (t_0, t_f].$$

$$\frac{\mathrm{d}\varphi_i(t)}{\mathrm{d}t} = \sum_{j=1}^{N_o} k_{2,i}^j \left[u\left(\xi^j, t\right) - z_{2,i}^j\right], \quad t \in (t_0, t_f], \quad i = 1, 2, \ldots, N_c. \tag{21}$$

In case of measurements at discrete moments of time (16), the differential equations (1), (8) will have the form

$$\frac{\partial u(x,t)}{\partial t} = a^2 \mathrm{div}\left(\mathrm{grad}\, u(x,t)\right) - \lambda_0 \left[u(x,t) - \theta\right] \tag{22}$$

$$+\sum_{i=1}^{N_c} \delta\left(x - \eta^i(t)\right) \sum_{j=1}^{N_o} k_{1,i}^j \left[\breve{u}_s^j - z_{1,i}^j\right], \quad x \in \Omega, \quad t \in (\tau_s, \tau_{s+1}], \quad s = 0, 1, \ldots, N_t,$$

$$\frac{\mathrm{d}\varphi_i(t)}{\mathrm{d}t} = \sum_{j=1}^{N_o} k_{2,i}^j \left[\breve{u}_s^j - z_{2,i}^j\right], \quad t \in (\tau_s, \tau_{s+1}], \quad s = 0, 1, \ldots, N_t, \quad i = 1, 2, \ldots, N_c. \tag{23}$$

In this case, the conditions of continuity of the processes under consideration must be met at the moments of measurement time:

$$u(x, \tau_s + 0) = u(x, \tau_s - 0), \quad \varphi_i(\tau_s + 0) = \varphi_i(\tau_s - 0), \tag{24}$$

$$i = 1, 2, \ldots, N_c, \quad s = 0, 1, \ldots, N_t.$$

The differential equation (20) is point-loaded due to the participation in it of the values of the sought function $u(x,t)$ at the measurement points $x = \xi^j$, $j = 1, 2, \ldots, N_o$ [15]. Note that the initial–boundary value problem (20), (2), (3) and the Cauchy problems (21), (9) must be solved simultaneously. The issues of existence and uniqueness of solutions of linearly loaded initial–boundary value problems and numerical methods for their solution were studied in such works as [15]–[19].

Let us combine the parameters $k_1 = ((k_{1,i}^j))$, $z_1 = ((z_{1,i}^j))$, $k_2 = ((k_{2,i}^j))$, $z_2 = ((z_{2,i}^j))$, $\xi = (\xi_1, \xi_2, \ldots, \xi_{N_o})$, $i = 1, 2, \ldots, N_c$, $j = 1, 2, \ldots, N_o$ into one $n = 4N_oN_c + 2N_o$, n-dimensional synthesized parameter vector $y = (k_1, z_1, k_2, z_2, \xi) \in \mathbb{R}^n$, which consists of: N_cN_o-dimensional vector k_1, N_cN_o-dimensional vector z_1, N_cN_o-dimensional vector k_2, N_cN_o-dimensional vector z_2, $2N_o$-dimensional vector ξ.

The problem requires determining the vector of parameters $y \in \mathbb{R}^n$ for which the constraints on the power and velocity of movement of sources (5) and (10) must be satisfied, and the objective functional

$$J(y) = \int_{\Theta} \int_{\mathcal{B}} I(y; b, \theta) \rho_{\mathcal{B}}(b)\, \rho_{\Theta}(\theta)\, db d\theta, \tag{25}$$

$$I(y; b, \theta) = \iint_{\Omega} \mu(x) \left[u(x, t_f) - U(x)\right]^2 dx + \varepsilon \left\|y - \hat{y}\right\|_{\mathbb{R}^n}^2. \tag{26}$$

must take the minimum possible value. Here $u(x,t) = u(x,t;y,b,\theta)$, $\eta^i(t) = \eta^i(t;y)$, $i = 1,2,\ldots,N_c$ are solutions of differential equations (20), (21) in case of continuous measurements and (22), (23) for discrete measurements of the process state for given feedback parameters $y = (k_1, z_1, k_2, z_2, \xi)$, initial condition $u(x,t_0) = b$, $\varphi_i(t_0) = \varphi_i^0$, $i = 1,2,\ldots,N_c$, at ambient temperature θ; ε, $\hat{y}$ are given regularization parameters of the functional of problem [14].

Let us note the following features of the resulting problem. Firstly, its specific features include the loading of the resulting differential equation and the participation of the Dirac function in it. Secondly, the value of the objective functional is determined not by a single solution of the initial–boundary value problem, but by a bundle of solutions, provided that the initial condition and the temperature of the external ambient take values from the corresponding sets $\mathcal{B}$ and Θ.

The next most important feature of the resulting control synthesis problem is its possible multi-extremality. This is due to the nonlinear participation in (17), (18), (19) of the parameters k_1, z_1, k_2, z_2, taking into account the implicit dependence of $u\left(\xi^j,t\right)$ themselves on these parameters. This feature should be taken into account when choosing a numerical optimization method. When conducting numerical experiments, we used the multi-start method, i.e., applying different initial values of the optimized parameters to the local optimization process.

In general, the resulting problem (20), (2), (3), (5), (21), (9), (25), (26) can be viewed as a finite-dimensional parametric optimal control problem for a distributed system with feedback, with respect to the vector $y \in \mathbb{R}^n$. In this setting, evaluating the objective functional for admissible feedback parameters requires solving initial-boundary value problems for coupled differential equations involving both partial and ordinary derivatives.

3 Formulas for the gradient of the problem functional. Necessary optimality conditions in problems (1)–(16) and (20)–(26).

For the numerical solution of the resulting parametric optimal control problem, first-order iterative optimization methods are proposed.

It can be readily verified that the objective functional of the original optimal control problem (1)-(12) is convex with respect to $q(t)$ and $\omega(t)$ for fixed trajectories $\eta(t)$ of the point sources. In contrast, the functional (25), (26) for the feedback-based problem (14), as follows from (20), (21), is generally non-convex with respect to the optimization parameters y. This non-convexity arises from the nonlinear dependence of the solution of the boundary value problem $u(x,t;y,b,\theta)$ on the parameter vector $y = (k_1, z_1, k_2, z_2, \xi)$, as noted above. Therefore, the optimality conditions derived in this section are local in nature. The expressions for the components of the functional gradient appearing in these conditions can be used in the numerical computation of locally optimal feedback parameters or for their local refinement. To obtain optimal parameter values, global optimization techniques may be combined with local refinement procedures based on first-order optimization methods that utilize the functional gradient formulas presented below.

We write constraints (5), (10) on control actions, using (17), (18), in the following form:

$$\underline{q_i} \le \sum_{j=1}^{N_o} k_{1,i}^j \left[u\left(\xi^j,t\right) - z_{1,i}^j \right] \le \overline{q_i}, \quad t \in [t_0, t_f], \quad i = 1,2,\ldots,N_c, \tag{27}$$

$$\underline{\omega_i} \le \sum_{j=1}^{N_o} k_{2,i}^j \left[u\left(\xi^j,t\right) - z_{2,i}^j \right] \le \overline{\omega_i}, \quad t \in [t_0, t_f], \quad i = 1,2,\ldots,N_c. \tag{28}$$

Let us introduce the following notation:

$$g_1^i(t;y) = \left|\tilde{g}_1^i(t;y)\right| - \frac{\overline{q_i} - \underline{q_i}}{2}, \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c,$$

$$\tilde{g}_1^i(t;y) = \sum_{j=1}^{N_o} k_{1,i}^j \left[u\left(\xi^j, t\right) - z_{1,i}^j\right] - \frac{\overline{q_i} + \underline{q_i}}{2},$$

$$g_2^i(t;y) = \left|\tilde{g}_2^i(t;y)\right| - \frac{\overline{\omega_i} - \underline{\omega_i}}{2}, \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c,$$

$$\tilde{g}_2^i(t;y) = \sum_{j=1}^{N_o} k_{2,i}^j \left[u\left(\xi^j, t\right) - z_{2,i}^j\right] - \frac{\overline{\omega_i} + \underline{\omega_i}}{2}.$$

Then constraints (27) and (28) can be rewritten in the following form:

$$g_1^i(t;y) \le 0, \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c, \tag{29}$$

$$g_2^i(t;y) \le 0, \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c. \tag{30}$$

To incorporate the constraints (5) and (10) into the parameter synthesis problem for y, we employ an exterior penalty method by augmenting the functional (25), (26) with a penalty term that accounts for violations of constraints (29) and (30):

$$J_{\mathcal{R}}(y) = \int\limits_{\Theta} \int\limits_{\mathcal{B}} I_{\mathcal{R}}(y; b, \theta) \rho_{\mathcal{B}}(b)\, \rho_{\Theta}(\theta)\, db d\theta, \tag{31}$$

$$I_{\mathcal{R}}(y; b, \theta) = \iint\limits_{\Omega} \mu(x) \left[u(x, t_f) - U(x)\right]^2 dx + \varepsilon \left\|y - \hat{y}\right\|_{\mathbb{R}^n}^2 + \mathcal{R}_1 G_1(y) + \mathcal{R}_2 G_2(y), \tag{32}$$

where

$$G_1(y) = \sum_{i=1}^{N_c} \int\limits_{t_0}^{t_f} \left[g_1^{i,+}(t;y)\right]^2 dt, \quad G_2(y) = \sum_{i=1}^{N_c} \int\limits_{t_0}^{t_f} \left[g_2^{i,+}(t;y)\right]^2 dt.$$

The functional (31) is minimized iteratively under the condition that the penalty parameters $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ approach $+\infty$. The notations $g_1^{i,+}(t;y)$, $g_2^{i,+}(t;y)$ mean that if $g_1^i(t;y) > 0$, $g_2^i(t;y) > 0$ and $g_1^{i,+}(t;y) = 0$, $g_2^{i,+}(t;y) = 0$, if $g_1^i(t;y) \le 0$, $g_2^i(t;y) \le 0$, $i = 1, 2, \ldots, N_c$, $j = 1, 2, \ldots, N_o$.

Next, we introduce the function $\operatorname{sgn}(\cdot)$, which takes the value -1 for negative arguments, 0 for a zero argument, and 1 for positive arguments.

In general, to synthesize local optimal values of the feedback parameters y, it is proposed to use iterative minimization methods with a combination of penalty function and gradient projection methods:

$$y^{\gamma+1} = \mathcal{P}_{(15)}\left[y^{\gamma} - \alpha^{\gamma} \operatorname{grad}_y J_{\mathcal{R}}(y^{\gamma})\right], \tag{33}$$

$$\alpha^{\gamma} = \arg\min_{\alpha \ge 0} J_{\mathcal{R}}\left(\mathcal{P}_{(15)}\left[y^{\gamma} - \alpha \operatorname{grad}_y J_{\mathcal{R}}(y^{\gamma})\right]\right), \quad \gamma = 0, 1, \ldots .$$

Here $\operatorname{grad}_y J_{\mathcal{R}}(y)$ is the gradient of the penalty functional (31), (32); α^{γ} is the step size in the direction of the antigradient of the functional $J_{\mathcal{R}}(y)$ at the γ-th iteration, obtained by one-dimensional minimization methods; $\mathcal{P}_{(15)}[\cdot]$ is the projection operator of a point onto the acceptable domain defined by conditions (15) [14].

It is clear that the key component required to implement procedure (33) is the expression for the gradient of the functional $\mathrm{grad}_y J_{\mathcal{R}}(y)$. The components of the gradient of the penalty functional with respect to the continuous feedback parameters are obtained from the following theorem.

Theorem 3.1. *The objective functional $J_{\mathcal{R}}(y)$ of the problem (20), (2), (3), (5), (21), (9), (10), (25), (26) is differentiated with respect to the synthesized parameters $y = (k_1, z_1, k_2, z_2, \xi)$ with continuous feedback (14). The components of the gradient of the functional are determined by the formulas*

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial k_{1,i}^j} = \int\limits_{\Theta}\int\limits_{\mathcal{B}}\Big\{-\int\limits_{t_0}^{t_f}\Big[\psi\left(\eta^i(t),t\right) - 2\mathcal{R}_1 sgn\left(\tilde{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\Big]\Big[u\left(\xi^j,t\right) - z_{1,i}^j\Big]dt \quad (34)$$

$$+2\varepsilon\left(k_{1,i}^j - \hat{k}_{1,i}^j\right)\Big\}\rho_{\mathcal{B}}(b)\,\rho_{\Theta}(\theta)\,dbd\theta,$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial z_{1,i}^j} = \int\limits_{\Theta}\int\limits_{\mathcal{B}}\Big\{\int\limits_{t_0}^{t_f}\Big[\psi\left(\eta^i(t),t\right) - 2\mathcal{R}_1 sgn\left(\tilde{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\Big]k_{1,i}^j dt \quad (35)$$

$$+2\varepsilon\left(z_{1,i}^j - \hat{z}_{1,i}^j\right)\Big\}\rho_{\mathcal{B}}(b)\,\rho_{\Theta}(\theta)\,dbd\theta,$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial k_{2,i}^j} = \int\limits_{\Theta}\int\limits_{\mathcal{B}}\Big\{-\int\limits_{t_0}^{t_f}\Big[\phi_i(t) - 2\mathcal{R}_2 sgn\left(\tilde{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\Big]\Big[u\left(\xi^j,t\right) - z_{2,i}^j\Big]dt \quad (36)$$

$$+2\varepsilon\left(k_{2,i}^j - \hat{k}_{2,i}^j\right)\Big\}\rho_{\mathcal{B}}(b)\,\rho_{\Theta}(\theta)\,dbd\theta,$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial z_{2,i}^j} = \int\limits_{\Theta}\int\limits_{\mathcal{B}}\Big\{\int\limits_{t_0}^{t_f}\Big[\phi_i(t) - 2\mathcal{R}_2 sgn\left(\tilde{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\Big]k_{2,i}^j dt \quad (37)$$

$$+2\varepsilon\left(z_{2,i}^j - \hat{z}_{2,i}^j\right)\Big\}\rho_{\mathcal{B}}(b)\,\rho_{\Theta}(\theta)\,dbd\theta,$$

$$\frac{\partial J_{\mathcal{R}}(y)}{\partial \xi_{\nu}^j} = \int\limits_{\Theta}\int\limits_{\mathcal{B}}\Big\{-\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\Big[\psi\left(\eta^i(t),t\right) - 2\mathcal{R}_1 sgn\left(\tilde{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\Big] \quad (38)$$

$$\times k_{1,i}^j\frac{\partial u(x,t)}{\partial x_{\nu}}\Big|_{x=\xi^j} + \Big[\phi_i(t) - 2\mathcal{R}_2 sgn\left(\tilde{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\Big]k_{2,i}^j\frac{\partial u(x,t)}{\partial x_{\nu}}\Big|_{x=\xi^j}dt$$

$$+2\varepsilon\left(\xi_{\nu}^j - \hat{\xi}_{\nu}^j\right)\Big\}\rho_{\mathcal{B}}(b)\,\rho_{\Theta}(\theta)\,dbd\theta,$$

$$i = 1,2,\ldots,N_c, \quad j = 1,2,\ldots,N_o, \quad \nu = 1,2.$$

The functions $\psi(x,t) = \psi(x,t;y,b,\theta)$, $\phi_i(t) = \phi_i(t;y)$, $i = 1,2,\ldots,N_c$ for the current parameter vector y, admissible initial condition $b \in \mathcal{B}$, and ambient temperature $\theta \in \Theta$ are obtained as solutions of the corresponding adjoint boundary value problem and associated Cauchy problems given below:

$$\frac{\partial \psi(x,t)}{\partial t} = -a^2 div\left(grad\,\psi(x,t)\right) + \lambda_0\psi(x,t) \quad (39)$$

$$-\sum_{j=1}^{N_o}\delta(x-\xi^j)\sum_{i=1}^{N_c}\left[\psi\left(\eta^i(t),t\right)-2\mathcal{R}_1 sgn\left(\tilde{g}_1^i(t;y)\right)g_1^{i,+}(t;y)\right]k_{1,i}^j$$

$$-\sum_{j=1}^{N_o}\delta(x-\xi^j)\sum_{i=1}^{N_c}\left[\phi_i(t)-2\mathcal{R}_2 sgn\left(\tilde{g}_2^i(t;y)\right)g_2^{i,+}(t;y)\right]k_{2,i}^j,$$

$$x\in\Omega,\quad t\in[t_0,t_f),$$

$$\psi(x,t_f)=-2\mu(x)\left[u(x,t_f)-U(x)\right],\quad x\in\Omega, \tag{40}$$

$$\frac{\partial\psi(x,t)}{\partial n}=\lambda\psi(x,t),\quad x\in\Gamma,\quad t\in[t_0,t_f), \tag{41}$$

$$\frac{d\phi_i(t)}{dt}=-\left.\frac{\partial\psi(x,t)}{\partial x_1}\right|_{x=\eta_1^i(t)}\left(\left.\frac{df_i(\gamma)}{d\gamma}\right|_{\gamma=\varphi_i(t)}\cos(\varphi_i(t))-f_i(\varphi_i(t))\sin(\varphi_i(t))\right) \tag{42}$$

$$\times\sum_{j=1}^{N_o}k_{1,i}^j\left[u\left(\xi^j,t\right)-z_{1,i}^j\right]$$

$$-\left.\frac{\partial\psi(x,t)}{\partial x_2}\right|_{x=\eta_1^i(t)}\left(\left.\frac{df_i(\gamma)}{d\gamma}\right|_{\gamma=\varphi_i(t)}\sin(\varphi_i(t))+f_i(\varphi_i(t))\cos(\varphi_i(t))\right),$$

$$\times\sum_{j=1}^{N_o}k_{1,i}^j\left[u\left(\xi^j,t\right)-z_{1,i}^j\right],\quad t\in[t_0,t_f),\quad i=1,2,\ldots,N_c,$$

$$\phi_i(t_f)=0,\quad i=1,2,\ldots,N_c. \tag{43}$$

To prove the theorem, we can use a well-known technology based on estimating the increment of the objective functional due to the increment of the optimized feedback parameters [2], [14]. It is easy to prove a similar theorem and obtain formulas for the gradient components of the objective functional in the case of discrete-time feedback of the form (16).

Let us formulate for the problem under consideration the necessary conditions for the optimality of the feedback parameters y^* in variational form [20].

Theorem 3.2. *For local optimality of the feedback parameters y^* in problem (20), (2), (3), (5), (21), (9), (10), (25), (26) with continuous feedback, it is necessary and sufficient that the following condition be satisfied:*

$$\langle grad_y J(y^*), y-y^*\rangle\le 0,$$

for all admissible values of the feedback parameters y that satisfy conditions (15), (29), (30).

4 Results of numerical experiments

We present results obtained from solving the following illustrative examples of plate heating control using continuous feedback. The parameter values and functions used in formulating problems (1)–(15) are listed below.

$$a^2=0.8,\quad\lambda_0=0.01,\quad\lambda=0.001,\quad\Omega=[0;1]\times[0;1],\quad t_0=0,\quad t_f=2,$$

$$\mu(x)=1,\ U(x)=10,\ x\in\Omega,\ \varepsilon=0.1,\ \varepsilon_3=0.0001,\ \varepsilon_4=0.0001,\ N_c=2,\ N_o=4,$$

$$\mathcal{B}=\{0.01i-0.03:\quad i=1,2,\ldots,N_{\mathcal{B}}\},\quad p_i^b=0.2,\quad i=1,2,\ldots,N_{\mathcal{B}},\quad N_{\mathcal{B}}=5,$$

$$\Theta = \{0.1j : \quad j = 1, 2, \ldots, N_\Theta\}, \quad p_j^\theta = 0.25, \quad j = 1, 2, \ldots, N_\Theta, \quad N_\Theta = 4,$$
$$\xi_1 \in \Omega_1 = [0.05;\ 0.45] \times [0.05;\ 0.45], \quad \xi_2 \in \Omega_2 = [0.05;\ 0.45] \times [0.55;\ 0.95],$$
$$\xi_3 \in \Omega_3 = [0.55;\ 0.95] \times [0.55;\ 0.95], \quad \xi_4 \in \Omega_4 = [0.55;\ 0.95] \times [0.05;\ 0.45],$$
$$0 \le q_1(t) \le 9, \quad 0 \le q_2(t) \le 7, \quad t \in [0;\ 2],$$
$$-8 \le \omega_1(t) \le 8, \quad -10 \le \omega_2(t) \le 10, \quad t \in [0;\ 2],$$
$$\varphi_1(0) = 0, \quad \varphi_2(0) = \pi,$$
$$r_1(t) = 0.4 + 0.03\cos(8\varphi_1(t)), \quad t \in [0;\ 2],$$
$$r_2(t) = 0.2 + 0.02\cos(8\varphi_2(t)), \quad t \in [0;\ 2],$$
$$\eta_1^1(t) = 0.5 + r_1(t)\cos(\varphi_1(t)), \quad \eta_2^1(t) = 0.5 + r_1(t)\sin(\varphi_1(t)), \quad t \in [0;\ 2],$$
$$\eta_1^2(t) = 0.5 + r_2(t)\cos(\varphi_2(t)), \quad \eta_2^2(t) = 0.5 + r_2(t)\sin(\varphi_2(t)), \quad t \in [0;\ 2].$$

Let us describe the general algorithm used to solve the problem of synthesizing the parameter vector y for the selected values of the penalty coefficients $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ and the regularization parameters ε, $\hat{y}$. When implementing procedure (33), at each iteration and for the current parameter values y^γ, the following steps are carried out for all admissible initial conditions $b_i \in \mathcal{B}$ and ambient temperatures $\theta_j \in \Theta$, $i = 1, 2, \ldots, N_\mathcal{B}$, $j = 1, 2, \ldots, N_\Theta$.

1. the initial–boundary value problem (20), (2), (3) and the Cauchy problems (21), (9) are solved;
2. the corresponding adjoint boundary value problem (39)–(41) and adjoint Cauchy problems (42), (43) are solved;
3. the components (34)–(38) of the gradient of the penalty functional (31), (32) are calculated;
4. one-dimensional minimization is carried out in the direction of the obtained antigradient vector of the functional according to the procedure (33).

These steps are repeated until the stopping criterion for the functional or for the argument y is met.

$$|J_\mathcal{R}\left(y^{\gamma+1}\right) - J_\mathcal{R}\left(y^\gamma\right)| \le \varepsilon_3,$$
$$\|y_\mathcal{R}^{\gamma+1} - y_\mathcal{R}^\gamma\|_{\mathbb{R}^n} \le \varepsilon_4.$$

At the next stage, the penalty parameter $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ was multiplied by five, and steps 1–4 were repeated until the stopping criteria for the functional value or the argument were satisfied at two consecutive stages with accuracies ε_3 and ε_4, respectively.

To solve two-dimensional direct (20), (2), (3) and adjoint (39)–(41) loaded boundary value problems, a modified scheme of the alternating directions method with steps in spatial variables $h_{x_1} = h_{x_2} = 0.01$ and in time variable $h_t = 0.01$ was used [21].

To solve the direct (21), (9) and adjoint (42), (43) Cauchy problems, a modified scheme of the Euler method with a step $h_t = 0.01$ was used.

To approximate the two-dimensional Dirac $\delta(\cdot)$-function, the following everywhere smooth (differentiable) trigonometric function was used [2], [7]:

$$\delta_\sigma(x;\tilde{x}) = \begin{cases} 0, & |x_1 - \tilde{x}_1| > \sigma_1 \quad \text{or} \quad |x_2 - \tilde{x}_2| > \sigma_2, \\ \prod_{i=1}^2 \frac{1}{2\sigma_i}\left[1 + \cos\left(\frac{x_i - \tilde{x}_i}{\sigma_i}\pi\right)\right], & |x_1 - \tilde{x}_1| \le \sigma_1 \quad \text{and} \quad |x_2 - \tilde{x}_2| \le \sigma_2. \end{cases}$$

In the experiments, the parameters $\sigma_i > 0$, $i = 1, 2$ in the function $\delta_\sigma(x;\eta)$ were chosen as $3h_{x_1}$ and $3h_{x_2}$, respectively, where h_{x_1}, h_{x_2} denote the grid step sizes used in the discretization of the two-dimensional domain Ω. This selection of the approximation function ensured sufficient smoothness of the functional $J_{\mathcal{R}}(y)$ with respect to the optimized current locations of the point heat sources $\eta(t)$ and measurement points ξ.

The comparison was made of the results of controlling sources both when they move along given trajectories and in the case when the sources do not move, and their locations are optimized [8], [22], [23].

First, we present the results of experiments obtained when controlling moving sources.

Table 1: Results of procedure (33) from initial points $y^{(1)}$ and $y^{(2)}$.

	γ	k_1		z_1		k_2		z_2		ξ		$J_{\mathcal{R}}(y^\gamma)$
$y^{(1)}$	0	-0.1264	-0.1234	8.9285	8.8694	-0.0421	-0.0454	8.8935	8.9836	0.1573	0.1485	8.1582
		-0.1315	-0.1276	8.9634	8.8354	-0.0372	-0.0418	8.8752	8.8725	0.3236	0.9135	
		-0.0854	-0.0745	8.9884	8.9475	-0.0229	-0.0195	8.8506	8.8427	0.8436	0.5658	
		-0.0924	-0.0691	8.7527	8.8556	-0.0218	-0.0198	8.8725	8.9748	0.9219	0.1218	
	20	-0.1783	-0.1793	9.9718	10.034	0.1713	0.1247	9.9348	9.8253	0.1744	0.1487	0.1427
		-0.1832	-0.1485	10.027	10.148	0.1265	0.0739	9.9805	10.135	0.1385	0.8125	
		-0.1138	-0.1651	9.9863	9.8554	-0.0259	-0.0193	9.8543	10.056	0.5982	0.7264	
		-0.1953	-0.1089	9.8452	10.183	-0.0734	-0.1514	10.125	9.9045	0.8289	0.3124	
	40^*	-0.2288	-0.2197	9.9995	9.9985	-0.2082	-0.1835	9.9978	9.9988	0.2589	0.2814	0.0001
		-0.2242	-0.2194	9.9993	9.9989	-0.2094	-0.2995	9.9983	9.9899	0.3297	0.7389	
		-0.2024	-0.1974	9.9952	9.9972	-0.2173	-0.2497	9.9984	9.9991	0.7436	0.6535	
		-0.1995	-0.1896	9.9978	9.9982	-0.2483	-0.2297	9.9978	9.9959	0.6598	0.2698	
$y^{(2)}$	0	-0.0141	-0.0267	9.2485	9.7467	-0.0049	-0.0068	9.8695	9.8757	0.2541	0.2467	12.182
		-0.0234	-0.0183	9.7334	9.9483	-0.0054	-0.0038	9.8542	9.7842	0.2534	0.7485	
		-0.0259	-0.0424	9.6259	9.8524	-0.0064	-0.0084	9.8449	9.8538	0.7669	0.7624	
		-0.0478	-0.0358	9.8478	9.9358	-0.0028	-0.0044	9.8156	9.8894	0.7478	0.2488	
	10	-0.1623	-0.1389	10.389	10.184	0.0234	0.0186	9.8835	9.9534	0.1334	0.1657	0.0885
		-0.1225	-0.1594	10.156	10.198	0.0289	0.0389	9.9535	9.9698	0.2504	0.7448	
		-0.1538	-0.1274	10.167	10.096	0.0274	0.0196	9.9315	9.9296	0.7922	0.7823	
		-0.1188	0.0047	10.113	10.154	0.0856	0.0375	9.9428	9.9077	0.8517	0.1728	
	20^*	-0.1989	-0.2035	9.9998	9.9999	-0.2182	-0.2295	9.9995	9.9927	0.1824	0.1645	0.0002
		-0.2295	-0.2148	9.9989	9.9984	-0.1937	-0.2259	9.9915	9.9939	0.2934	0.6762	
		-0.2355	-0.2084	9.9993	9.9959	-0.2136	-0.2095	9.9994	9.9939	0.7438	0.6744	
		-0.1999	-0.2185	9.9942	9.9959	-0.2037	-0.2284	9.9985	9.9961	0.7935	0.2795	

Table 1 presents the results obtained at different iterations for two distinct initial points $y^{(1)}$ and $y^{(2)}$. As was mentioned above about the possible multi-extremality of the objective functional, the optimization results obtained from different initial points differ in arguments, although the difference in functionality is not significant. Here it is also necessary to take into account (as other specially conducted numerical experiments have shown) that the functional of the problem has a significantly ravine structure.

Fig. 1 shows the movements of the controlled sources along the given trajectories (thin dashed lines) and the positions of the measurement points for the initial vector $y^0 = y^{(1)}$, as well as for the synthesized optimal vector of feedback parameters $y^{(1),*}$.

Fig. 2 shows graphs of the power of heat sources (on the left) and the velocity of their movement along given trajectories (on the right) for the initial and synthesized optimal feedback parameters $y^0 = y^{(1)}$ and $y^* = y^{(1),*}$. Dashed lines are graphs for the initial vector $y^{(1)}$, solid lines are graphs for the synthesized vectors $y^{(1),*}$ obtained as a result of optimization.

Numerical experiments were carried out in which the exact values of the observed process states at the measurement points $u\left(\xi^j, t\right)$, $j = 1, 2, \ldots, N_o$, with optimal feedback parameters $y^* = y^{(1),*}$ were noised by random error according to the formula

$$\breve{u}_j^\chi(t; y^*) = u\left(\xi^j, t; y^*\right)\left[1 + \chi_j(t)\right], \quad t \in [t_0, t_f], \quad j = 1, 2, \ldots, N_o.$$

Here $u\left(\xi^j, t; y^*\right)$ denotes the computed solution of the direct initial–boundary value problem at the point ξ^j, $j = 1, 2, \ldots, N_o$ for $t \in [t_0, t_f]$. The term $\chi_j(t)$ for $t \in [t_0, t_f]$ is a random

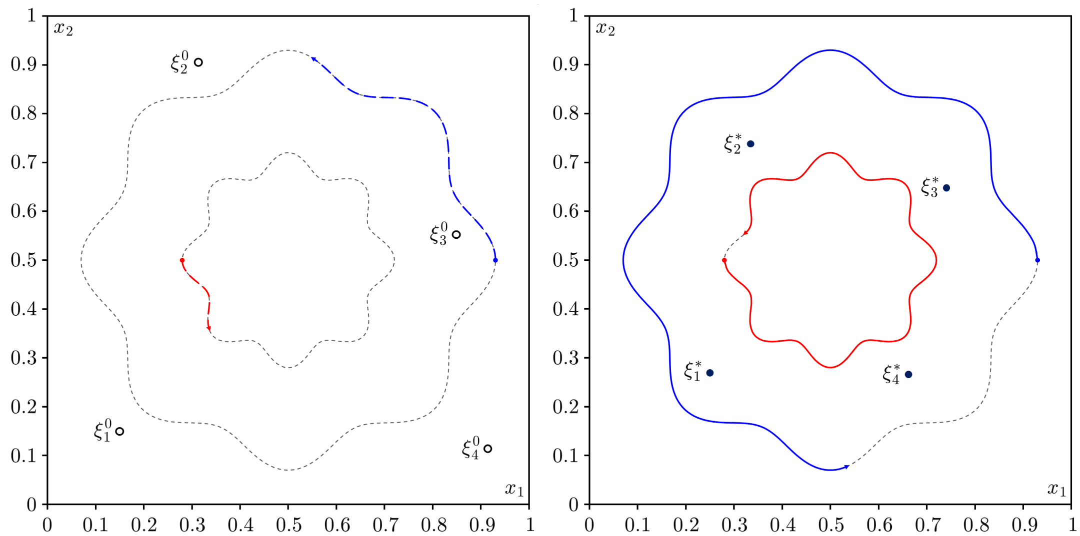

Figure 1: Graphs of the movement of heat sources along given trajectories and the positions of control points for the initial vector y^0 and the synthesized optimal vector y^*.

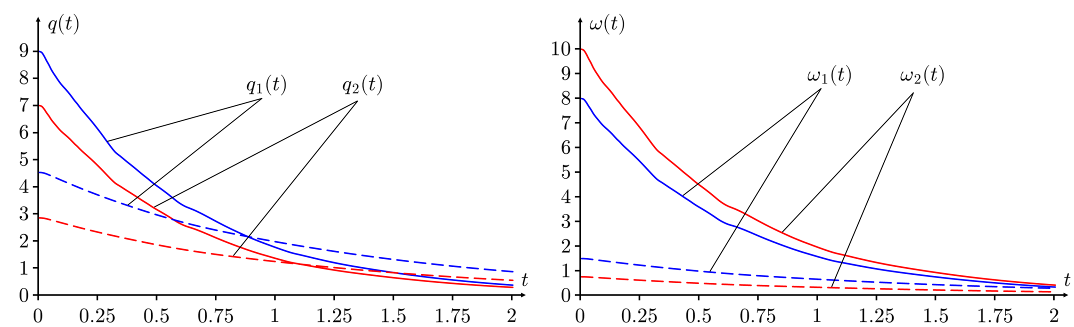

Figure 2: Graphs of the power of heat sources (left) and the speed of their movement (right) for the initial $y^{(1)}$ (dashed lines) and synthesized $y^{(1),*}$ (solid lines) vectors.

variable uniformly distributed on the interval $[-\zeta, \zeta]$, where ζ specifies the maximum noise level $j = 1, 2, \ldots, N_o$. In the experiments carried out, the values of ζ were chosen equal to 0.01; 0.03; 0.05, which corresponds to a measurement error of 1%, 3% and 5% from the exact (calculated) values of the measured quantities.

Fig. 3 shows the graphs of the function

$$\Phi\left(t; y^*\right) = \sum_{j=1}^{N_\Theta} \sum_{i=1}^{N_\mathcal{B}} \left\{ \iint\limits_{\Omega} \mu\left(x\right) \left[u\left(x, t; y^*, b_i, \theta_j\right) - U\left(x\right)\right]^2 dx \right\} p_i^b p_j^\theta,$$

where $u\left(x, t; y^*, b, \theta\right)$ denotes the solution of the initial–boundary value problem (20), (2), (3) corresponding to the optimal feedback parameter vector $y^* = y^{(1),*}$ obtained from the numerical solution of the problem. For clarity, graphs for different levels of interference in the state measurements are shown at different time intervals and scales along the ordinate axis.

An important indicator of the quality of control of the heating process with the used feedback parameters is the function $\Phi\left(t; y^*\right)$, which numerically characterizes the result of process control on average for all possible values of external influences. Fig. 3 shows graphs of the function $\Phi\left(t; y^*\right)$, obtained with optimal feedback parameters and measurement error levels equal to 0% (no error), 1%, 3%, and 5%. These graphs show that the quality of control over the stabilization process corresponds to the magnitude of the measurement error of the process state. As can be seen from Fig. 3, the function $\Phi(t; y^*)$, and consequently the heating process itself, is quite stable to measurement errors, and this stability is maintained when controlling the process at $t \geq 2$. In Fig. 3, a) due to the small scale, the values of the function $\Phi(t; y^*)$ at $t \in [0; 1.8]$ at different noise levels are practically indistinguishable. Therefore, in Fig. 3, b) for $t \in [1.8; 2.5]$ the scale for the axis of function values is increased.

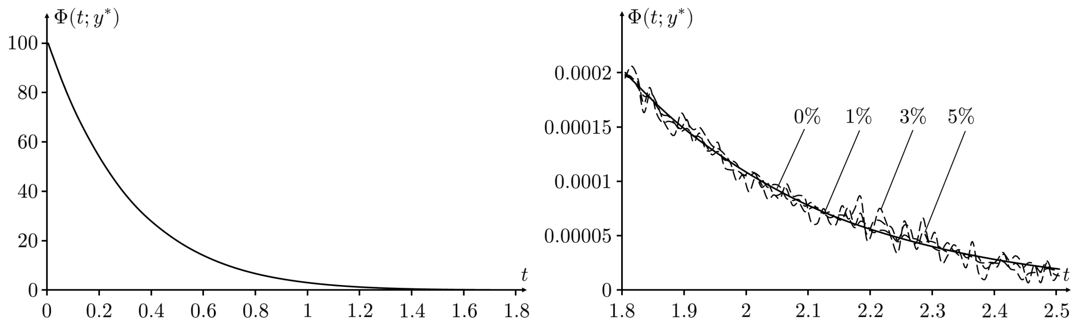

Figure 3: Graphs of the values of the function $\Phi(t; y^*)$ in $t \in [0; 2.5]$ for the feedback parameters $y^{(1),*}$ at the noise level of 1%, 3%, 5%.

Of interest is the following comparison between the results of the feedback control problem for sources moving along prescribed trajectories and those of a control problem in which the sources are stationary, while their locations, along with the positions of the control points on the plate, are treated as optimization variables.

For the objective comparison, a condition for the possible total amount of heat during heating has been added to the problem

$$\sum_{i=1}^{N_c} \int_{t_0}^{t_f} q_i(t; \tilde{y}^*)\, dt \leq \overline{Q}.$$

Here the value $\overline{Q} = 10.62$ is equal to the used value of heat obtained when solving the problem of controlling moving sources.

Table 2: Initial and obtained values of the optimized parameters and functionality.

γ	k_1		z_1		ξ		η		$J_{\mathcal{R}}(\check{y}^{\gamma})$
0	-0.0342	-0.0487	9.3458	9.9383	0.2455	0.2248	0.3587	0.4785	9.57844
	-0.0187	-0.0248	9.6892	9.9248	0.3584	0.5487	0.8759	0.4758	
	-0.0247	-0.0319	9.5482	9.6518	0.6587	0.5978			
	-0.0459	-0.0238	9.6587	9.9124	0.7658	0.2759			
$*$	-0.2285	-0.2297	9.9879	9.9842	0.1495	0.2834	0.4933	0.2184	0.1374
	-0.2194	-0.2239	9.9926	9.9972	0.2094	0.7294	0.6223	0.7520	
	-0.2289	-0.2128	9.9892	9.9918	0.6295	0.8534			
	-0.2295	-0.2184	9.9896	9.9925	0.7834	0.1795			

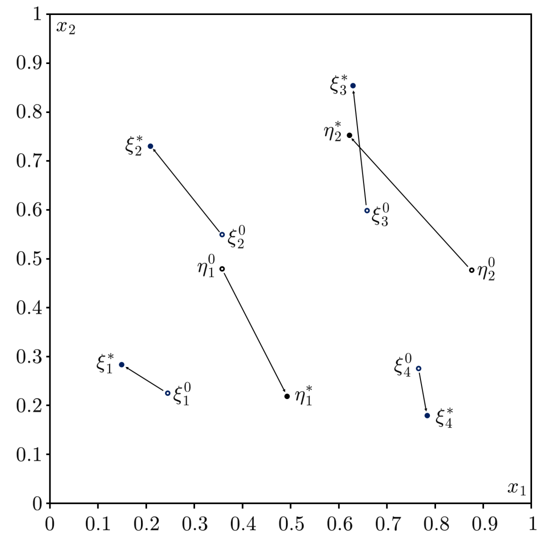

Figure 4: Initial and optimal locations of sources and measuring sensors.

The Table 2 and the Fig. 4 show the obtained results of solving the problem with optimized locations of sources and measurement points. From Table 2 it is evident that the obtained value of the objective functional significantly exceeds the optimal value of the functional in the problem with moving sources, given in Table 1. The Fig. 5 shows the initial and optimal locations for heat sources and measuring sensors. We note that we used the results of works [22], [23] in which variants of the problem of controlling point sources during plate heating were investigated.

Fig. 4 shows graphs of the power of sources obtained with the initial parameter vector $\breve{y}^0$ and the synthesized optimal feedback parameters $\breve{y}^*$ for stationary point sources. The dotted lines correspond to the graphs for the initial vector $\breve{y}^0$, the solid lines correspond to the synthesized vector $\breve{y}^*$.

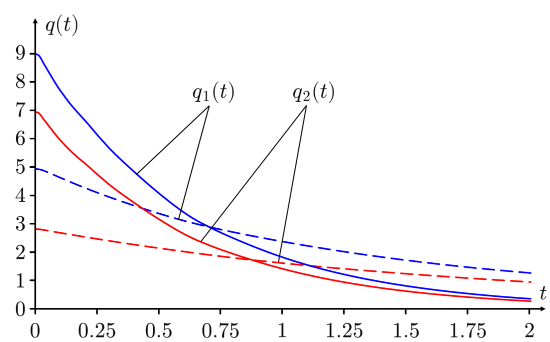

Figure 5: Graphs of the powers of stationary sources corresponding to the initial parameter vector $\breve{y}^0$ (dashed lines) and the synthesized vector $\breve{y}^*$ (solid lines).

Conclusion

An approach to feedback control of motion along given trajectories is proposed. To form the current values of heating of the power plate and control a actions on the movement of sources, linear dependencies are proposed from the results of the temperature at the points of placement of measuring sensors on the results of the measurements taken. The differentiability of the functional with respect to the feedback parameters involved in these dependencies and the locations of the measuring sensors is shown. Formulas for the gradient of the functional with respect to the synthesized parameters are obtained. The formulas allow the use of efficient numerical methods of first-order optimization and available standard software packages to solve the problem of synthesis of control of lumped sources.

It should be noted that the proposed approach to synthesis leads to the problem of parametric optimal control of a process described by loaded differential equations with ordinary and partial derivatives.

The proposed approach to the control of lumped sources with feedback can be used in automatic control systems and regulation of lumped sources of other technological processes and technical objects described by other form and types of boundary value problems.

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Received 18.12.2025 , Accepted 15.06.2026, Available online 30.06.2026.