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WELL-POSEDNESS AND EXPONENTIAL STABILITY FOR A POROUS-ELASTIC SYSTEM IN THERMOELASTICITY OF TYPE III WITH DISTRIBUTED DELAY

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Abstract In this article, we study the well-posedness and asymptotic behaviour of solutions one-dimensional porous-elastic system in thermoelasticity of type III with distributed delay term. We first give the well-posedness of the system by using semigroup method and Lumer-Philips theorem. Then, by using the energy method and construct some Lyapunov functionals, we obtain the exponential decay of the solution for the case of equal speed of wave propagation.

Key words: Porous-elastic system, thermoelasticity of type III, distributed delay, well-posedness, exponential stability.

AMS Mathematics Subject Classification: 35B40, 93D20, 35L05, 35L70.

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1 Introduction

In this article, we investigate the well-posedness and exponential stability for a porous-elastic system in thermoelasticity of type III with distributed delay term, under the initial and boundary conditions, which has the form

$$\begin{aligned}
&\rho\omega_{tt} - \mu\omega_{xx} - b\varphi_x = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\
&J\varphi_{tt} - \delta\varphi_{xx} + b\omega_x + \xi\varphi + \beta\theta_x = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\
&\alpha\theta_{tt} - \delta\theta_{xx} + \beta\varphi_{ttx} - k\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s)\theta_{txx}(x,t-s)ds = 0, \quad (x,t)\in(0,1)\times(0,\infty),\\
&\omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad \varphi(x,0) = \varphi_0(x), \quad x \in (0,1), \qquad (1)\\
&\varphi_t(x,0) = \varphi_1(x), \quad \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad x \in (0,1),\\
&\omega(0,t) = \omega(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad t \in (0,\infty),\\
&\theta_{tx}(x,-t) = f_0(x,t), \quad (x,t) \in (0,1)\times(0,\tau_2),
\end{aligned}$$

where $\omega = \omega(x,t)$, $\varphi = \varphi(x,t)$ and $\theta = \theta(x,t)$ are the displacement of the solid elastic material, the volume fraction, and the difference temperature, respectively. The parameter ρ is the mass density and J equals to the product of the equilibrated inertia by the mass density. The coefficients μ, b, δ, ξ, β, α, and k are positive constants, such that

$$\mu/\rho = \delta/J, \qquad (2)$$

and

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$$\mu\xi > b^2, \tag{3}$$

and $g : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, with τ_1 and τ_2 are two real numbers satisfying $0 \le \tau_1 < \tau_2$. The initial data ω_0, ω_1, φ_0, φ_1, θ_0, θ_1, and f_0 belongs to the suitable functional space.

In [1], Lacheheb et al. considered the system (1) without distributed delay term. The authors established the well-posedness result and proved that the system is exponentially stable under condition (2), (3) and a polynomial decay when the wave-propagation speeds are different. As for the previous results and developments of porous-elastic problems, they have stated and summarized in great detail in [1]. The readers, for a better understanding of present work, are strongly recommended to [1] and the reference therein (e.g. [2]-[9]).

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences. Introducing the delay term makes the problem different from those considered in the literatures. It has been established that voluntary introduction of delay can benefit the control (see [10]). On the other hand, it may not only destabilize a system which is asymptotically stable in the absence of delay but may also lead to ill-posedness (see [11, 12] and the references therein). Therefore, the issue of well-posedness and the stability result of systems with delay are of practical and theoretical importance. In recent years, the control of partial differential equations with time delay effects has become an active area of research (e.g. [13]–[20]). Nicaise and Pignotti [19] considered the wave equation with liner frictional damping and internal distributed delay

$$\rho u_{tt} - \triangle u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_1(s)\, u_t(t-s)\, ds = 0,$$

in $\Omega \times (0,1)$, with initial and mixed Dirichlet-Neumann boundary conditions and a is a suitable function. They obtained exponential decay of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds < \mu_1.$$

The authors also obtained the same result when the distributed delay acted on the part of the boundary. Kafini et al. [15] considered the following Timoshenko-type system of thermoelasticity of type III with distributive delay

$$\begin{aligned}
&\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x = 0,\\
&\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_{tx} = 0,\\
&\rho_3 \theta_{tt} - \delta\theta_{xx} - k\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s)\theta_{txx}(x, t-s)ds + \gamma\varphi_{tx} = 0,
\end{aligned}$$

where $\tau_1 < \tau_2$ are non-negative constants such that $g : [\tau_1, \tau_2] \to \mathbb{R}^+$ represents distributive time delay. They proved an exponential decay in the case of equal wave speeds and a polynomial decay result in the case of nonequal wave speeds with smooth initial data. Recently, Khochemane et al. [16] considered the following one-dimensional porous-elastic system with distributed delay term acting on the porous equation

$$\begin{aligned}
&\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad x \in (0,1), \quad t > 0,\\
&J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\phi_t(t-s)ds = 0, \quad x \in (0,1), \quad t > 0.
\end{aligned}$$

Under suitable assumptions on the weight of distributed delay, the authors established the well-posedness result and show that the dissipation given by this complementary control stabilizes exponentially the system for the case of equal speeds of wave propagation.

Motivated by the above results, we establish the well-posedness result and prove that the system is exponentially stable for the case of equal speed of wave propagation. The paper is organized as follows. In Section 1, we present preliminaries and main results. In Section 2, we use the semi-group method to prove the existence and uniqueness of the solutions. In Section 3, we use the energy method to prove the exponential stability result under the conditions (2) and (3).

2 Preliminaries and main results

To exhibit the dissipative nature of the problem (1), we introduce some new variables

$$u = \omega_t, \quad \phi = \varphi_t.$$

As in [21], we introduce the new variable

$$z(x,\rho,s,t) = \theta_{tx}(x,t-\rho s), \quad x \in (0,1), \quad \rho \in (0,1), \quad s \in (\tau_1,\tau_2), \quad t > 0.$$

A simple differentiation shows that z satisfies

$$sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \quad x \in (0,1), \quad \rho \in (0,1), \quad s \in (\tau_1,\tau_2), \quad t > 0.$$

Then system (1) takes the form

$$\begin{aligned}
&\rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \quad (x,t) \in (0,1) \times (0,\infty),\\
&J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_{tx} = 0, \quad (x,t) \in (0,1) \times (0,\infty),\\
&\alpha\theta_{tt} - \delta\theta_{xx} + \beta\phi_{tx} - k\theta_{txx} - \int_{\tau_1}^{\tau_2} g(s) z_x(x,1,s,t)ds = 0, \quad (x,t) \in (0,1)\times(0,\infty),\\
&sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \quad (x,\rho,s,t) \in (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty),\\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \phi(x,0) = \phi_0(x), \quad x \in (0,1),\\
&\phi_t(x,0) = \phi_1(x), \quad \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad x \in (0,1),\\
&u(0,t) = u(1,t) = \phi_x(0,t) = \phi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad t \in (0,\infty),\\
&z(x,\rho,s,0) = f_0(x,\rho s), \quad (x,\rho,s) \in (0,1) \times (0,1) \times (\tau_1,\tau_2).
\end{aligned} \tag{4}$$

Concerning the weight of the delay, we assume that

$$\int_{\tau_1}^{\tau_2} |g(s)|\, ds < k. \tag{5}$$

From equation $(4)_2$ and the boundary conditions, we easily verify that

$$\frac{d^2}{dt^2}\int_0^1 \phi(x,t)\,dx + \frac{\xi}{J}\int_0^1 \phi(x,t)\,dx = 0.$$

We introduce

$$\bar\phi(x,t) = \phi(x,t) - \left(\int_0^1 \phi_0(x)\,dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right) - \sqrt{\frac{J}{\xi}}\left(\int_0^1 \phi_1(x)\,dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right).$$

We know that $\left(u,\bar{\phi},\theta,z\right)$ satisfies the boundary conditions, and more importantly

$$\int_0^1 \bar{\phi}\,(x,t)\,dx=0,\ \forall t\geq 0.$$

Hence, the use of Poincaré inequality for $\bar{\phi}$ is justified. In what follows, we will work with $\bar{\phi}$. For convenience, we write ϕ.

From now on, we let $U=(u,v,\phi,\psi,\theta,\vartheta,z)^T$, where $v=u_t$, $\psi=\phi_t$ and $\vartheta=\theta_t$. System (4) can be written as an evolutionary equation

$$U'(t)=\mathcal{A}U(t),\quad t>0,\quad U(0)=U_0=(u_0,u_1,\phi_0,\phi_1,\theta_0,\theta_1,f_0)^T. \tag{6}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}U=\begin{pmatrix} v\\ \dfrac{\mu}{\rho}u_{xx}+\dfrac{b}{\rho}\phi_x\\ \psi\\ \dfrac{\delta}{J}\phi_{xx}-\dfrac{b}{J}u_x-\dfrac{\xi}{J}\phi-\dfrac{\beta}{J}\vartheta_x\\ \vartheta\\ \dfrac{\delta}{\alpha}\theta_{xx}-\dfrac{\beta}{\alpha}\psi_x+\dfrac{k}{\alpha}\vartheta_{xx}+\dfrac{1}{\alpha}\int_{\tau_1}^{\tau_2}g\left(s\right)z_x(x,1,s,t)ds\\ -\dfrac{1}{s}z_\rho(x,\rho,s,t)\end{pmatrix}.$$

We consider the following space $H_*^1(0,1)=H^1(0,1)\cap L_*^2(0,1)$, where $L_*^2(0,1)=\{w\in L^2(0,1),\int_0^1 w(x)dx=0\}$, and the energy space

$$\mathcal{H}=H_0^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L_*^2(0,1)\times H_0^1(0,1)\times L^2(0,1)\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),$$

equipped with the inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_0^1[\rho v\widetilde{v}+\mu u_x\widetilde{u}_x+\xi\phi\widetilde{\phi}+J\psi\widetilde{\psi}+\delta\phi_x\widetilde{\phi}_x+\delta\theta_x\widetilde{\theta}_x+\alpha\vartheta\widetilde{\vartheta}+bu_x\widetilde{\phi}+b\phi\widetilde{u}_x]dx\\ &+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|g\left(s\right)\right|z\left(x,\rho,s\right)\widetilde{z}\left(x,\rho,s\right)dsd\rho dx. \end{aligned}\tag{7}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{c} U\in\mathcal{H}\mid u\in H^2(0,1)\cap H_0^1(0,1),\quad v\in H_0^1(0,1),\quad \phi\in H_*^2(0,1)\cap H_*^1(0,1),\\ \psi\in H_*^1(0,1),\quad \vartheta\in H_0^1(0,1),\quad \delta\theta+\left(k+\int_{\tau_1}^{\tau_2}g(s)e^{-s}ds\right)\vartheta\in H^2(0,1),\\ z,z_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\end{array}\right\},$$

where $H_*^2(0,1)=\{w\in H^2(0,1);\ w_x(0)=w_x(1)=0\}$. Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$. We give the following well-posedness result of problem (6).

Theorem 2.1. *Let $U_0\in\mathcal{H}$ and assume that (5) hold. Then, there exists a unique solution $U\in C(\mathbb{R}^+;\mathcal{H})$ of problem (6). Moreover, if $U_0\in D(\mathcal{A})$, then $U\in C(\mathbb{R}^+;D(\mathcal{A}))\cap C^1(\mathbb{R}^+;\mathcal{H})$.*

To state our decay result, we introduce the following energy functional:

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho u_t^2 + J\phi_t^2 + \alpha\theta_t^2 + \mu u_x^2 + \delta\phi_x^2 + \delta\theta_x^2 + \xi\phi^2 + 2b\phi u_x\right] dx \\ &+ \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|g(s)\right| z^2(x,\rho,s,t)\, ds d\rho dx. \end{aligned} \tag{8}$$

We have the following exponentially stable result.

Theorem 2.2. *Let (u,ϕ,θ,z) be the solution of the system (4) and we assume that (2), (5) hold. Then, the solution (u,ϕ,θ,z) decays exponentially, i.e. there exist two positive constants k_0 and k_1 such that*

$$E(t) \le k_0 e^{-k_1 t}, \quad \forall t \ge 0. \tag{9}$$

3 Well-posedness

In this section, we give the proof of the well-posedness of problem (4) by making use of Lumer-Philips theorem [22, 23].

Proof. (of **Theorem** 2.1) To prove the well-posedness result, it suffices to show that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective.

Step 1. $\mathcal{A}$ is dissipative.

For any $U = (u,v,\phi,\psi,\theta,\vartheta,z)^T \in D(\mathcal{A})$, by using the inner product and integrating by parts, we can imply that

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -k\int_0^1 \vartheta_x^2 dx + \int_0^1 \vartheta\int_{\tau_1}^{\tau_2} g(s) z_x(x,1,s,t) ds dx \\ &- \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |g(s)| z(x,\rho,s,t) z_\rho(x,\rho,s,t) ds d\rho dx. \end{aligned} \tag{10}$$

The last two terms of the right side of (10) can be estimated as follows

$$\begin{aligned} &\int_0^1 \vartheta\int_{\tau_1}^{\tau_2} g(s) z_x(x,1,s,t) ds dx = -\int_0^1 \vartheta_x\int_{\tau_1}^{\tau_2} g(s) z(x,1,s,t) ds dx \\ \le\ & \frac{1}{2}\int_{\tau_1}^{\tau_2} |g(s)| ds \int_0^1 \vartheta_x^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |g(s)| z^2(x,1,s,t) ds dx, \end{aligned} \tag{11}$$

$$\begin{aligned} &-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |g(s) z(x,\rho,s,t) z_\rho(x,\rho,s,t) ds d\rho dx \\ =\ & \frac{1}{2}\int_{\tau_1}^{\tau_2} |g(s)| ds \int_0^1 \vartheta_x^2 dx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |g(s)| z^2(x,1,s,t) ds dx. \end{aligned} \tag{12}$$

Substituting (11) and (12) into (10), and using (5), we obtain

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le -\left(k - \int_{\tau_1}^{\tau_2} |g(s)| ds\right)\int_0^1 \vartheta_x^2 dx \le 0.$$

Hence, the operator $\mathcal{A}$ is dissipative.

Step 2. $Id-\mathcal{A}$ is surjective.

To prove that the operator $Id-\mathcal{A}$ is surjective, that is, for any $F=(f_1,...,f_7)\in\mathcal{H}$, there exists $U=(u,v,\phi,\psi,\theta,\vartheta,z)^T\in D(\mathcal{A})$ satisfying

$$(Id-\mathcal{A})U=F, \tag{13}$$

which is equivalent to

$$\begin{aligned} &u-v=f_1,\quad \rho v-\mu u_{xx}-b\phi_x=\rho f_2,\quad \phi-\psi=f_3,\\ &J\psi-\delta\phi_{xx}+bu_x+\xi\phi+\beta\vartheta_x=Jf_4,\quad \theta-\vartheta=f_5,\\ &\alpha\vartheta-\delta\theta_{xx}+\beta\psi_x-k\vartheta_{xx}-\int_{\tau_1}^{\tau_2}g\,(s)\,z_x(x,1,s,t)ds=\alpha f_6,\\ &sz(x,\rho,s,t)+z_\rho(x,\rho,s,t)=sf_7. \end{aligned} \tag{14}$$

We note that the last equation in (14) with $z(x,0,s,t)=\vartheta_x(x,t)$, has a unique solution

$$z(x,\rho,s,t)=\vartheta_x(x,t)e^{-\rho s}+se^{-\rho s}\int_0^\rho e^{\delta s}f_7(x,\delta,s)d\delta. \tag{15}$$

$(14)_1$, $(14)_3$ and $(14)_5$ give

$$v=u-f_1,\quad \psi=\phi-f_3,\quad \vartheta=\theta-f_5. \tag{16}$$

Inserting (16) into $(14)_2$, $(14)_4$ and $(14)_6$, we get

$$\rho u-\mu u_{xx}-b\phi_x=h_1,\ (J+\xi)\phi-\delta\phi_{xx}+bu_x+\beta\theta_x=h_2,\ \alpha\theta-\mu_1\theta_{xx}+\beta\phi_x=h_3, \tag{17}$$

where

$$\begin{aligned} h_1&=\rho f_1+\rho f_2,\quad h_2=Jf_3+Jf_4+\beta\partial_x f_5,\\ h_3&=\alpha f_5+\alpha f_6+\beta\partial_x f_3-\left(k+\int_{\tau_1}^{\tau_2}g\,(s)\,e^{-s}ds\right)\partial_{xx}f_5\\ &\quad+\int_{\tau_1}^{\tau_2}g\,(s)\,se^{-s}\int_0^1 e^{\delta s}\partial_x f_7\,(x,\delta,s)\,d\delta ds,\\ \mu_1&=\delta+k+\int_{\tau_1}^{\tau_2}g\,(s)\,e^{-s}ds. \end{aligned}$$

In order to solve (17), we consider the following variational formulation

$$B\big((u,\phi,\theta)^T,(\tilde{u},\tilde{\phi},\widetilde{\theta})^T\big)=L(\tilde{u},\tilde{\phi},\widetilde{\theta})^T, \tag{18}$$

where $B:[H_0^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1)]^2\to\mathbb{R}$ is the bilinear form given by

$$\begin{aligned} B\big((u,\phi,\theta)^T,(\tilde{u},\tilde{\phi},\widetilde{\theta})^T\big)&=\rho\int_0^1 u\tilde{u}dx+\mu\int_0^1 u_x\tilde{u}_xdx+(J+\xi)\int_0^1\phi\tilde{\phi}dx\\ +\delta\int_0^1\phi_x\tilde{\phi}_xdx+\alpha\int_0^1\theta\widetilde{\theta}dx&+\mu_1\int_0^1\theta_x\widetilde{\theta}_xdx+b\int_0^1(\phi\tilde{u}_x+u_x\tilde{\phi})dx+\beta\int_0^1(\theta_x\tilde{\phi}+\phi_x\widetilde{\theta})dx, \end{aligned}$$

and $L:[H_0^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1)]\to\mathbb{R}$ is the linear form defined by

$$L(\tilde{u},\tilde{\phi},\widetilde{\theta})^T=\int_0^1 h_1\tilde{u}dx+\int_0^1 h_2\tilde{\phi}dx+\int_0^1 h_3\widetilde{\theta}dx.$$

Now, for $V=H_0^1(0,1)\times H_*^1(0,1)\times H_0^1(0,1)$ equipped with the norm

$$\|(u,\phi,\theta)\|_V^2 = \|u\|^2 + \|\phi\|^2 + \|\theta\|^2 + \|u_x\|^2 + \|\phi_x\|^2 + \|\theta_x\|^2,$$

then, we have

$$\begin{aligned} B\big((u,\phi,\theta)^T,(u,\phi,\theta)^T\big) &= \rho\int_0^1 u^2dx + \delta\int_0^1 \phi_x^2dx + \alpha\int_0^1 \theta^2dx \\ + \quad \mu_1\int_0^1 \theta_x^2dx &+ (J+\xi)\int_0^1 \phi^2dx + \mu\int_0^1 u_x^2dx + 2b\int_0^1 \phi u_x dx. \end{aligned}$$

On the other hand, we can write

$$\begin{aligned} &\mu u_x^2 + 2b\phi u_x + \xi\phi^2 \\ &= \frac{1}{2}\left[\mu\left(u_x+\frac{b}{\mu}\phi\right)^2 + \xi\left(\phi+\frac{b}{\xi}u_x\right)^2 + \left(\mu-\frac{b^2}{\xi}\right)u_x^2 + \left(\xi-\frac{b^2}{\mu}\right)\phi^2\right], \end{aligned}$$

since $\mu\xi > b^2$, we deduce that

$$\mu u_x^2 + 2b\phi u_x + \xi\phi^2 > \frac{1}{2}\left[\left(\mu-\frac{b^2}{\xi}\right)u_x^2 + \left(\xi-\frac{b^2}{\mu}\right)\phi^2\right],$$

then, for some $M_0 > 0$,

$$B\left((u,\phi,\theta)^T,(u,\phi,\theta)^T\right) \geq M_0\,\|(u,\phi,\theta)\|_V^2\,.$$

Thus, B is coercive. On the other hand, we can easily show, using Cauchy-Schwarz inequality, that B and L are continuous. Applying the Lax-Milgram Lemma, we deduce that for all $(\tilde{u},\tilde{\phi},\tilde{\theta})^T \in V$, problem (18) admits a unique solution $(u,\phi,\theta)^T \in V$. Applying the classical elliptic regularity, it follows from (17) that

$$u \in H^2(0,1), \quad \phi \in H_*^2(0,1), \quad \delta\theta + \left(k + \int_{\tau_1}^{\tau_2} g(s)e^{-s}ds\right)\vartheta \in H^2(0,1).$$

Therefore, the operator $Id - \mathcal{A}$ is surjective. Consequently, the result of Theorem 2.1 follows from the Lumer-Philips theorem. □

4 Exponential stability

In this section, we prove the exponential decay result in Theorem 2.2. It will be achieved by using the energy method. To achieve our goal, we need the following lemmas.

Lemma 4.1. *Let (u,ϕ,θ,z) be the solution of (4) and assume (5) holds. Then the energy functional, defined by (8) satisfies*

$$E'(t) \leq -\left(k - \int_{\tau_1}^{\tau_2} |g(s)|ds\right)\int_0^1 \theta_{tx}^2 dx \leq 0, \quad \forall t \geq 0. \tag{19}$$

Proof. Multiplying $(4)_1$, $(4)_2$ and $(4)_3$ by u_t, ϕ_t and θ_t, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left[\rho u_t^2+\mu u_x^2+\xi\phi^2+J\phi_t^2+\delta\phi_x^2+\alpha\theta_t^2+\delta\theta_x^2+2bu_x\phi\right]dx\\
=\;&-k\int_0^1\theta_{tx}^2dx+\int_0^1\theta_t\int_{\tau_1}^{\tau_2}g(s)z_x(x,1,s,t)dsdx. \qquad (20)
\end{aligned}$$

Multiplying $(4)_4$ by $|g(s)|z$, integrating the product over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, and recall that $z(x,0,s,t)=\theta_{tx}(x,t)$, yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|g\left(s\right)\right|z^2(x,\rho,s,t)dsd\rho dx\\
=\;&\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|g\left(s\right)\right|\theta_{tx}^2dsdx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|g\left(s\right)\right|z^2(x,1,s,t)dsdx. \qquad (21)
\end{aligned}$$

A combination of (20) and (21) gives

$$\begin{aligned}
E'(t)\;=\;&-k\int_0^1\theta_{tx}^2dx+\int_0^1\theta_t\int_{\tau_1}^{\tau_2}g\left(s\right)z_x(x,1,s,t)dsdx\\
&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|g(s)|\theta_{tx}^2dsdx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|g(s)|z^2(x,1,s,t)dsdx. \qquad (22)
\end{aligned}$$

Meanwhile, using Young's inequality, we have

$$\begin{aligned}
&\int_0^1\theta_t\int_{\tau_1}^{\tau_2}g(s)z_x(x,1,s,t)dsdx=-\int_0^1\theta_{tx}\int_{\tau_1}^{\tau_2}g(s)z(x,1,s,t)dsdx\\
\leq\;&\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|g(s)|ds\right)\int_0^1\theta_{tx}^2dx+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|g(s)|z^2(x,1,s,t)dsdx. \qquad (23)
\end{aligned}$$

Simple substitution of (23) into (22) and using (5) give (19), which concludes the proof. □

Remark 1. *Note that $E(t)$ is non-negative. In fact, by considering*

$$\begin{aligned}
&\mu u_x^2+2b\phi u_x+\xi\phi^2\\
&=\frac{1}{2}\left[\mu\left(u_x+\frac{b}{\mu}\phi\right)^2+\xi\left(\phi+\frac{b}{\xi}u_x\right)^2+\left(\mu-\frac{b^2}{\xi}\right)u_x^2+\left(\xi-\frac{b^2}{\mu}\right)\phi^2\right],
\end{aligned}$$

and using (3), we get $\mu u_x^2+2b\phi u_x+\xi\phi^2>0$. Consequently, it follows that $E(t)>0$.

Next, in order to construct a Lyapunov functional equivalent to the energy, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 4.2. *Let (u,ϕ,θ,z) be the solution of (5). Then the functional*

$$L_1\left(t\right)=-\rho\int_0^1 uu_tdx,$$

satisfies the estimate

$$L_1'(t)\leq-\rho\int_0^1 u_t^2dx+\frac{3\mu}{2}\int_0^1 u_x^2dx+\frac{b^2}{2\mu}\int_0^1\phi_x^2dx. \qquad (24)$$

Proof. By differentiating $L_1(t)$ with respect to t, then exploiting the first equation in (4), and integrating by parts, we obtain

$$L_1'(t) = -\rho\int_0^1 u_t^2dx + \mu\int_0^1 u_x^2dx + b\int_0^1 u_x\phi dx.$$

Using Young's and Poincaré inequalities, estimate (24) is established. □

Lemma 4.3. *Let (u,ϕ,θ,z) be the solution of (4). Then the functional*

$$L_2(t) = \frac{\mu}{\rho}\int_0^1 \phi_t u_x dx + \frac{\delta}{J}\int_0^1 \phi_x u_t dx,$$

satisfies the estimate

$$L_2'(t)\leq -\frac{\mu b}{2\rho J}\int_0^1 u_x^2dx+\frac{\mu\xi^2}{\rho J b}\int_0^1\phi^2dx+\frac{\delta b}{J\rho}\int_0^1\phi_x^2dx+\frac{\mu\beta^2}{\rho J b}\int_0^1\theta_{tx}^2dx. \tag{25}$$

Proof. By differentiating $L_2(t)$ with respect to t, then exploiting the first and the second equations in (4), and integrating by parts, we obtain

$$\begin{aligned} L_2'(t) &= -\frac{\mu b}{\rho J}\int_0^1 u_x^2dx - \frac{\mu\xi}{\rho J}\int_0^1 \phi u_x dx + \frac{\delta b}{J\rho}\int_0^1\phi_x^2dx \\ &- \frac{\mu\beta}{\rho J}\int_0^1 \theta_{tx}u_xdx + \left(\frac{\delta}{J}-\frac{\mu}{\rho}\right)\int_0^1 \phi_{tx}u_tdx. \end{aligned} \tag{26}$$

Using Young's inequality, we obtain

$$-\frac{\mu\xi}{\rho J}\int_0^1 \phi u_x dx \leq \frac{\mu b}{4\rho J}\int_0^1 u_x^2dx + \frac{\mu\xi^2}{\rho J b}\int_0^1\phi^2dx, \tag{27}$$

$$-\frac{\mu\beta}{\rho J}\int_0^1 \theta_{tx}u_x dx \leq \frac{\mu b}{4\rho J}\int_0^1 u_x^2dx + \frac{\mu\beta^2}{\rho J b}\int_0^1\theta_{tx}^2dx. \tag{28}$$

Substituting (27) and (28) in (26) and using (2), we get (25). □

Lemma 4.4. *Let (u,ϕ,θ,z) be the solution of (4). Then the functional*

$$L_3(t) = J\int_0^1 \phi\phi_t dx - \frac{\rho b}{\mu}\int_0^1 u_t\left(\int_0^x \phi(y)dy\right)dx,$$

satisfies, for any $\varepsilon_1>0$, the estimate

$$L_3'(t)\leq -\frac{\delta}{2}\int_0^1\phi_x^2dx - \gamma_1\int_0^1\phi^2dx + \left(J+\frac{\rho^2b^2}{4\varepsilon_1\mu^2}\right)\int_0^1\phi_t^2dx + \varepsilon_1\int_0^1 u_t^2dx + \frac{\beta^2}{2\delta}\int_0^1\theta_t^2dx, \tag{29}$$

where $\gamma_1 = \xi - b^2/\mu > 0$.

Proof. By differentiating $L_3(t)$ with respect to t, then exploiting the first and the second equations in (4), and integrating by parts, we obtain

$$L_3'(t) = J\int_0^1\phi_t^2dx + J\int_0^1\phi\phi_{tt}dx - \frac{\rho b}{\mu}\int_0^1 u_{tt}\left(\int_0^x\phi(y)dy\right)dx - \frac{\rho b}{\mu}\int_0^1 u_t\left(\int_0^x\phi_t(y)dy\right)dx$$

$$= J\int_0^1 \phi_t^2 dx - \delta\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + \beta\int_0^1 \theta_t\phi_x dx - \frac{\rho b}{\mu}\int_0^1 u_t\left(\int_0^x \phi_t(y)dy\right)dx. \quad (30)$$

By using Young's inequality, we have

$$\beta\int_0^1 \theta_t\phi_x dx \le \frac{\beta^2}{2\delta}\int_0^1 \theta_t^2 dx + \frac{\delta}{2}\int_0^1 \phi_x^2 dx, \quad (31)$$

using Young's and Cauchy–Schwarz inequalities with $\varepsilon_1 > 0$, we get

$$-\frac{\rho b}{\mu}\int_0^1 u_t\left(\int_0^x \phi_t(y)\,dy\right)dx \le \varepsilon_1\int_0^1 u_t^2 dx + \frac{\rho^2 b^2}{4\varepsilon_1\mu^2}\int_0^1 \phi_t^2 dx. \quad (32)$$

Inserting (31) and (32) in (30), we obtain (29). □

Lemma 4.5. *Let (u, ϕ, θ, z) be the solution of (4). Then the functional*

$$L_4(t) = -\alpha\int_0^1 \theta_t\left(\int_0^x \phi_t(y)\,dy\right)dx,$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$L_4'(t) \le -\frac{\beta}{4}\int_0^1 \phi_t^2 dx + \varepsilon_2\int_0^1 \phi_x^2 dx + \varepsilon_2\int_0^1 \phi^2 dx + \frac{\delta^2}{\beta}\int_0^1 \theta_x^2 dx + C_1(\varepsilon_2)\int_0^1 \theta_t^2 dx$$
$$+ \frac{k^2}{\beta}\int_0^1 \theta_{tx}^2 dx + \varepsilon_2\int_0^1 u_x^2 dx + \frac{1}{\beta}\int_{\tau_1}^{\tau_2}|g(s)|ds\int_0^1\int_{\tau_1}^{\tau_2}|g(s)|z^2(x,1,s,t)dsdx, \quad (33)$$

where $C_1(\varepsilon_2) = \dfrac{\beta\alpha}{J} + \dfrac{b^2\alpha^2}{4\varepsilon_2 J^2} + \dfrac{\delta^2\alpha^2}{4\varepsilon_2 J^2} + \dfrac{\xi^2\alpha^2}{4\varepsilon_2 J^2}$.

Proof. By differentiating $L_4(t)$ with respect to t, then exploiting the second and the third equations in (4), and integrating by parts, we obtain

$$L_4'(t) = -\beta\int_0^1 \phi_t^2 dx + \delta\int_0^1 \theta_x\phi_t dx + k\int_0^1 \theta_{tx}\phi_t dx + \int_0^1 \phi_t\int_{\tau_1}^{\tau_2} g(s)z(x,1,s,t)dsdx$$
$$+ \frac{\beta\alpha}{J}\int_0^1 \theta_t^2 dx - \frac{\delta\alpha}{J}\int_0^1 \theta_t\phi_x dx + \frac{b\alpha}{J}\int_0^1 \theta_t u dx + \frac{\xi\alpha}{J}\int_0^1 \theta_t\left(\int_0^x \phi dy\right)dx. \quad (34)$$

Using Young's, Cauchy–Schwarz and Poincaré inequalities,

$$\delta\int_0^1 \theta_x\phi_t dx \le \frac{\beta}{4}\int_0^1 \phi_t^2 dx + \frac{\delta^2}{\beta}\int_0^1 \theta_x^2 dx, \quad (35)$$

$$k\int_0^1 \theta_{tx}\phi_t dx \le \frac{\beta}{4}\int_0^1 \phi_t^2 dx + \frac{k^2}{\beta}\int_0^1 \theta_{tx}^2 dx, \quad (36)$$

$$\int_0^1 \phi_t\int_{\tau_1}^{\tau_2} g(s)z(x,1,s,t)dsdx \le \frac{\beta}{4}\int_0^1 \phi_t^2 dx + \frac{1}{\beta}\int_{\tau_1}^{\tau_2}|g(s)|ds$$
$$\times\int_0^1\int_{\tau_1}^{\tau_2}|g(s)|z^2(x,1,s,t)dsdx, \quad (37)$$

$$-\frac{\delta\alpha}{J}\int_0^1 \theta_t\phi_x dx \le \frac{\delta^2\alpha^2}{4\varepsilon_2 J^2}\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 \phi_x^2 dx, \quad (38)$$

$$\frac{b\alpha}{J}\int_0^1 \theta_t u dx \le \frac{b^2\alpha^2}{4\varepsilon_2 J^2}\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 u_x^2 dx, \quad (39)$$

$$\frac{\xi\alpha}{J}\int_0^1 \theta_t\left(\int_0^x \phi dy\right)dx \le \frac{\xi^2\alpha^2}{4\varepsilon_2 J^2}\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 \phi^2 dx, \quad (40)$$

where $\varepsilon_2 > 0$. Estimate (33) follows by substituting (35)-(40) into (34). □

Lemma 4.6. *Let (u, ϕ, θ, z) be the solution of (4). Then the functional*

$$L_5(t) = \alpha \int_0^1 \theta\theta_t dx + \frac{k}{2} \int_0^1 \theta_x^2 dx + \beta \int_0^1 \phi_x \theta dx,$$

satisfies, for any $\varepsilon_3 > 0$, the estimate

$$\begin{aligned} L_5'(t) \ \leq \ & -\frac{\delta}{2} \int_0^1 \theta_x^2 dx + \left(\alpha + \frac{\beta^2}{4\varepsilon_3}\right) \int_0^1 \theta_t^2 dx + \varepsilon_3 \int_0^1 \phi_x^2 dx \\ & + \frac{1}{2\delta} \int_{\tau_1}^{\tau_2} |g(s)| ds \int_0^1 \int_{\tau_1}^{\tau_2} |g(s)| z^2(x, 1, s, t) ds dx. \end{aligned} \tag{41}$$

Proof. By differentiating $L_5(t)$ with respect to t, using the equation $(4)_3$ and integrating by parts, we obtain

$$L_5'(t) = -\delta \int_0^1 \theta_x^2 dx + \alpha \int_0^1 \theta_t^2 dx + \beta \int_0^1 \phi_x \theta_t dx - \int_0^1 \theta_x \int_{\tau_1}^{\tau_2} g(s) z(x, 1, s, t) ds dx. \tag{42}$$

Using Young's and Cauchy–Schwarz inequalities with $\varepsilon_3 > 0$, we get

$$\beta \int_0^1 \phi_x \theta_t dx \ \leq \ \frac{\beta^2}{4\varepsilon_3} \int_0^1 \theta_t^2 dx + \varepsilon_3 \int_0^1 \phi_x^2 dx, \tag{43}$$

$$\begin{aligned} -\int_0^1 \theta_x \int_{\tau_1}^{\tau_2} g(s) z(x, 1, s, t) ds dx \ \leq \ & \frac{\delta}{2} \int_0^1 \theta_x^2 dx + \frac{1}{2\delta} \int_{\tau_1}^{\tau_2} |g(s)| ds \\ & \times \int_0^1 \int_{\tau_1}^{\tau_2} |g(s)| z^2(x, 1, s, t) ds dx. \end{aligned} \tag{44}$$

Substituting (43) and (44) in (42), we get (41). □

Lemma 4.7. *Let (u, ϕ, θ, z) be the solution of (4). Then the functional*

$$L_6(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |g(s)| z^2 (x, \rho, s, t) \, ds d\rho dx,$$

satisfies, for some positive constant n_1, the following estimate

$$\begin{aligned} L_6'(t) \ \leq \ & -n_1 \int_0^1 \int_{\tau_1}^{\tau_2} |g(s)| z^2(x, 1, s, t) ds dx + k \int_0^1 \theta_{tx}^2 dx \\ & -n_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |g(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned} \tag{45}$$

Proof. By differentiating $L_6(t)$ with respect to t, and using the equation $(4)_4$, we obtain

$$\begin{aligned} L_6'(t) \ = \ & -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |g(s)| z(x, \rho, s, t) z_\rho(x, \rho, s, t) ds d\rho dx \\ = \ & -\frac{d}{d\rho} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho} |g(s)| z^2(x, \rho, s, t) ds d\rho dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |g(s)| z^2(x, \rho, s, t) ds d\rho dx \\ = \ & - \int_0^1 \int_{\tau_1}^{\tau_2} |g(s)| \left[e^{-s} z^2(x, 1, s, t) - z^2(x, 0, s, t) \right] ds dx \\ & - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |g(s)| z^2(x, \rho, s, t) ds d\rho dx. \end{aligned}$$

Using the fact that $z(x,0,s,t)=\theta_{tx}$ and $e^{-s}\le e^{-s\rho}\le 1$, for all $0<\rho<1$ we obtain

$$\begin{aligned} L_6'(t) \;\le\; & -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|g(s)|z^2(x,1,s,t)dsdx+\int_{\tau_1}^{\tau_2}|g(s)|ds\int_0^1\theta_{tx}^2dx \\ & -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s}|g(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s}\le -e^{-\tau_2}$, for all $s\in[\tau_1,\tau_2]$. Finally, setting $n_1=e^{-\tau_2}$ and recalling (5), we obtain (45). □

Proof. (of Theorem 2.2) We define the Lyapunov functional L as follows

$$L(t)=NE(t)+L_1(t)+\sum_{i=2}^{6}N_iL_i(t), \tag{46}$$

where N, N_2, N_3, N_4, N_5 and N_6 are positive constants to be determined properly.

By differentiating (46) and recalling (19), (24), (25), (29), (33), (41), (45) and using of $\int_0^1\theta_t^2dx\le\int_0^1\theta_{tx}^2dx$, we arrive at

$$\begin{aligned} & L'(t)\le \\ & -[\rho-\varepsilon_1N_3]\int_0^1 u_t^2dx-\left[\frac{\beta}{4}N_4-\left(J+\frac{\rho^2b^2}{4\varepsilon_1\mu^2}\right)N_3\right]\int_0^1\phi_t^2dx \\ & -\left[N\left(k-\int_{\tau_1}^{\tau_2}|g(s)|ds\right)-\frac{\mu\beta^2}{\rho Jb}N_2-\frac{\beta^2}{2\delta}N_3-\left(C_1(\varepsilon_2)+\frac{k^2}{\beta}\right)N_4\right. \\ & \left.-\left(\alpha+\frac{\beta^2}{4\varepsilon_3}\right)N_5-kN_6\right]\int_0^1\theta_{tx}^2dx-\left[\frac{\mu b}{2\rho J}N_2-\frac{3\mu}{2}-\varepsilon_2N_4\right]\int_0^1u_x^2dx \\ & -\left[\frac{\delta}{2}N_3-\frac{b^2}{2\mu}-\frac{\delta b}{J\rho}N_2-\varepsilon_2N_4-\varepsilon_3N_5\right]\int_0^1\phi_x^2dx \\ & -\left[\frac{\delta}{2}N_5-\frac{\delta^2}{\beta}N_4\right]\int_0^1\theta_x^2dx-\left[\gamma_1N_3-\frac{\mu\xi^2}{\rho Jb}N_2-\varepsilon_2N_4\right]\int_0^1\phi^2dx \\ & -\left[n_1N_6-\left(\frac{N_4}{\beta}+\frac{N_5}{2\delta}\right)\int_{\tau_1}^{\tau_2}|g(s)|ds\right]\int_0^1\int_{\tau_1}^{\tau_2}|g(s)|z^2(x,1,s,t)dsdx \\ & -n_1N_6\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|g(s)|z^2(x,\rho,s,t)dsd\rho dx. \end{aligned} \tag{47}$$

At this point, we need to choose our constants carefully. First, we take $N_5=(4\delta/\beta)N_4$. We set $\varepsilon_1=\rho/(2N_3)$, then we choose N_2 large enough such that $(N_2\mu b)/(2\rho J)-3\mu/2>0$. Next, we select N_3 large so that $(\delta N_3)/2-b^2/(2\mu)-(\delta bN_2)/(J\rho)>0$, $\gamma_1N_3-(\mu\xi^2N_2)/(\rho Jb)>0$. Now, we choose N_4 large so that $(\beta N_4)/4-N_3(J+\rho^2b^2)/(4\varepsilon_1\mu^2)>0$. After that, we select ε_2, ε_3 small enough so that $(\mu bN_2)/(2\rho J)-3\mu/2-\varepsilon_2N_4>0$, $\gamma_1N_3-(\mu\xi^2N_2)(\rho Jb)-\varepsilon_2N_4>0$, and $(\delta N_3)/2-b^2/(2\mu)-(\delta bN_2)/(J\rho)-\varepsilon_2N_4-\varepsilon_3N_5>0$. Furthermore, we can take N_6 sufficiently large such that $n_1N_6-(N_4/\beta+N_5/(2\delta))\int_{\tau_1}^{\tau_2}|g(s)|ds>0$. Finally, we choose N large enough such that $\gamma>0$, where

$\gamma = N\left(k - \int_{\tau_1}^{\tau_2} |g(s)|ds\right) - (\mu\beta^2 N_2)/(\rho J b) - (\beta^2 N_3)/(2\delta) - (C_1(\varepsilon_2) + k^2/\beta)N_4 - (\alpha + \beta^2/(4\varepsilon_3))N_5 - kN_6$.

Thus, we obtain that there exists a positive constant η_0 such that (47) yields

$$L'(t) \leq -\eta_0 E(t), \ \forall t \geq 0. \tag{48}$$

On the hand, it is not hard to see that $L(t) \sim E(t)$, i.e. there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{49}$$

Combining (48) and (49), we obtain that

$$L'(t) \leq -k_1 L(t), \quad \forall t \geq 0, \tag{50}$$

for the positive constant $k_1 = \eta_0/\beta_2$. A simple integration of (50) over $(0, t)$ gives

$$L(t) \leq L(0)e^{-k_1 t}, \quad \forall t \geq 0. \tag{51}$$

Finally, by combining (49) and (51) we obtain (9) with $k_0 = \beta_2 E(0)/\beta_1$, which completes the proof. □

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