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INTERPOLATION BY CUBIC TENSION SPLINES WITH CONVEXITY INHERITANCE

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Abstract We consider the problem of interpolation of a convex function by tension splines with inheritance of the convexity condition. Previously, we developed the near-optimal algorithm for automatically selecting tension parameters when interpolating a convex function; now we propose to use it in the problem of piecewise convex interpolation. Using numerical examples, we show that tension parameters determined on subdomains when constructing separate splines can be used in a global tension spline construction over all data, which ensures the required convexity on subdomains.

Key words: convex interpolation, cubic tension spline, algorithm, sufficient conditions of convexity.

AMS Mathematics Subject Classification: 65D05, 65D07, 65D17.

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1 Introduction

The paper is devoted to the process of representing a given data set by a smooth function that exactly reproduces the given values and preserves such geometric property of the data as convexity. The problem of convex interpolation is very relevant in a wide range of applications in science, engineering and computer graphics. Other geometric properties such as positivity or monotonicity may also be of interest. The problem with inheritance such properties are commonly referred to as the problem of shape preserving interpolation.

The shape preserving interpolation problem refers to the requirement that the interpolant S or some of its derivative $S^{(k)}$ be nonnegative if the interpolated function f or, respectively, its derivative $f^{(k)}$ is nonnegative. The nonnegativity of the kth derivative is traditionally called k-monotonicity or monotonicity of order k (sometimes the term k-convexity is used alternatively). For small values of k, there are special names for k-monotonicity: nonnegativity (positivity) or sign-constancy for $k = 0$, monotonicity for $k = 1$ and convexity for $k = 2$.

In this paper, we limit ourselves to consider only problems reduced to convex interpolation problems, i.e., the requirement that the second derivative of the interpolant S be nonnegative if the derivative f'' of the interpolated function f is nonnegative. In addition, the problem of piecewise convex interpolation is considered, i.e., if the

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interpolated function has different signs of convexity in different subdomains, it is necessary to provide the interpolant with the required sign of the second derivative in these domains.

Although classical methods of interpolation by Lagrange and Hermite polynomials are still used today, spline methods prevail in practical problems. Polynomial interpolation often leads to sharp oscillations (see Runge's example [1, p.11]), but this is particularly undesirable in most applied problems, such as design, when practical considerations require that the data have some geometric properties, such as a constant sign of some derivative of the function representing the original data. Interpolation using Lagrange polynomials can often lead to quite acceptable results, but in some cases the behavior will be very different from what is expected. In most practical applications, oscillations are not acceptable at all.

Practical interest in splines arose after Holladay's paper [2] with the established property of minimum curvature of interpolating cubic splines

$$\int_a^b |\sigma''(x)|^2 \, dx. \tag{1}$$

The most attractive property of the cubic spline is that the function is represented by a particular cubic polynomial at each point and has smoothness C^2 at the original data points, so due to the discontinuities of the third derivatives at the junctions of the polynomials, the flexibility of the interpolant increases.

Although spline methods have now become the main tools for solving most problems of function approximation, cubic splines are not ideal; it is in the problems of shape preserving interpolation that we often have to abandon their use, because in the general case of interpolation cubic spline does not provide inheritance of the necessary geometric properties of the original function. Undesirable oscillations may occur if the original data are not "dense" enough even in the problem of approximation of arbitrarily smooth functions.

Since interpolation by ordinary cubic splines may not preserve the shape, Schweikert [3] in 1966 modified the scheme by replacing cubic polynomial splines $s(x)$ by functions minimising the integral

$$\int_a^b \left[|\sigma''(x)|^2 + |\rho\, \sigma'(x)|^2 \right] dx. \tag{2}$$

At $\rho = 0$ the function $s(x)$ is again a cubic spline; at $\rho \to \infty$ the spline $s(x)$ tends to a linear spline. It is thus ρ a tension parameter, and by choosing it large enough, one can always achieve preservation of the desired shape, such as monotonicity or convexity, everywhere or locally.

Many papers have been published on Schweikert tension splines, in which methods for selecting values of the tension parameters are proposed. However, the fact that the method uses exponential functions can be seen as a drawback. Therefore, in 1974, Späth [4] proposed the idea of using rational functions in tension interpolation methods.

Later, other types of tension splines were also introduced. All these constructions are based on the idea of generalizing the construction of a conventional cubic spline by introducing so-called shape (tension) parameters. These parameters allow to provide sufficient tension of the spline links in critical regions, which allows to control the shape of the spline. The appropriate choice of parameters leads to suppression of unwanted waves at a small deviation from the classical spline.

Previously, we have shown [5] that most of the constructions of cubic tension splines known in the literature can be represented in a unified form. We proposed a general algorithm for automatic selection of control parameters close to the optimal ones.

The selection is close to optimal because even a small decrease in them leads to violation of the convexity of the interpolant, and an increase leads to excessive tension.

It is known that cubic splines of class C^2 and their generalizations possess the property of locality, despite their global character. Changing any value of the interpolated function changes the interpolation spline over the entire solution interval. However, noticeable changes in the interpolant occur only in the neighborhood of the changed value, and at a distance the changes are quite insignificant.

In this paper, we show how this locality property can be used in the problem of piecewise convex interpolation with convexity sign inheritance in subdomains with data of the same convexity sign. We give numerical examples demonstrating the applicability of our algorithm to achieve local convexity.

2 Cubic Tension Splines

Let the values $f_i = f(x_i)$, $i = 0, \ldots, n$, of some function $f(x)$ be known at the knots of the mesh

$$\Delta : a = x_0 < x_1 < \ldots < x_n = b.$$

In addition, let the values of the first derivative $f'(a) = f'_a$, $f'(b) = f'_b$ at the ends of the interval $[a, b]$ be known also. For convenience, we expand the mesh Δ with additional multiple knots $x_{-1} = x_0$ and $x_{n+1} = x_n$. Denote the second divided differences of this function with respect to the knots x_{i-1}, x_i, x_{i+1} by $\delta_i = [x_{i-1}, x_i, x_{i+1}]f$, $i = 0, \ldots, n$.

We call data convex if following conditions hold

$$\delta_i \geq 0, \qquad i = 0, \ldots, n. \tag{3}$$

It is evident that for any convex function $f(x)$ the conditions (3) hold.

Consider a problem of construction of C^2-spline $S(x)$ that is convex at the interval $[a, b]$ and interpolates some convex data at the knots of Δ. Certainly we require $S'(x)$ takes given values at the endpoints:

$$S'(a) = f'_a, \qquad S'(b) = f'_b. \tag{4}$$

The cubic spline on every interval $[x_i, x_{i+1}]$ is a polynomial, it can be represented by its values and the values of the second derivative at the endpoints:

$$S(x) = \sigma(x) + \phi(1-t)h_i^2 M_i + \phi(t)h_i^2 M_{i+1}, \tag{5}$$

where

$$\sigma(x) = (1-t)f_i + tf_{i+1}, \;\; \phi(t) = \frac{t^3 - t}{6}\,, \;\; t = \frac{x - x_i}{h_i}\,, \;\; h_i = x_{i+1} - x_i, \;\; M_i = S''(x_i).$$

The continuity conditions of C^2-spline's derivatives at the knots of the mesh lead to a linear system of equations for finding the unknown values of moments M_i. The term $\sigma(x)$ ensures linear interpolation therefore it is evident, that possible "troubles" in behavior of ordinary cubic spline appear only due to behavior of summands with ϕ in (5).

Schweikert [3] modified the scheme by replacing cubic polynomials on each interval by minimization of (2) i.e. by the elements of the space $span\{1; x; \sinh px; \cosh px\}$ instead of the space $span\{1; x; x^2; x^3\}$. Späth [6] offered to differ the tension parameters p_i on each interval $[x_i, x_{i+1}]$.

Such a spline, which we refer to as hyperbolic spline, can be represented in the form

$$S(x) = \sigma(x) + \phi(p_i, 1-t)\, h_i^2 M_i + \phi(p_i, t)\, h_i^2 M_{i+1}, \tag{6}$$

where the function

$$\phi(p,t) = \frac{1}{p^2}\left(\frac{\sinh pt}{\sinh p} - t\right), \tag{7}$$

as a function of t, replaces the function $\phi(t) = (t^3 - t)/6$ in (5) and has an additional parameter p to control the tension of the spline. When $p_i = 0$, the spline reduces to a cubic spline, whereas if $p_i \to \infty$, the spline approaches a linear polynomial.

Following [5, 7] we consider the generalized cubic spline of the form

$$S(x) = \sigma(x) + \phi(q_i, 1-t)\, h_i^2 M_i + \phi(p_{i+1}, t)\, h_i^2 M_{i+1}, \tag{8}$$

with $\phi(p,t) \in C^2[0,1]$ and two control parameters on each interval. Parameters q_i and p_{i+1} control the tension near the knots x_i and x_{i+1} to the right and to the left correspondingly. The expressions for derivatives of generalized spline on the interval $[x_i, x_{i+1}]$ have form

$$S'(x) = f[x_i, x_{i+1}] - \phi'(q_i, 1-t) h_i M_i + \phi'(p_{i+1}, t) h_i M_{i+1}, \tag{9}$$

$$S''(x) = \phi''(q_i, 1-t) M_i + \phi''(p_{i+1}, t) M_{i+1}. \tag{10}$$

The moments M_i and M_{i+1} are the second derivatives of spline $S(x)$ in the corresponding interval endpoints. Certainly the interpolation conditions and a sense of M_i and M_{i+1} give the constraints for all values of the parameter p:

$$\phi(p,0) = \phi(p,1) = \phi''(p,0) = 0, \quad \phi''(p,1) = 1, \tag{11}$$

where differentiation is done with respect to t. Besides we should require that

$$\lim_{p\to 0} \phi(p,t) = (t^3 - t)/6, \quad \lim_{p\to\infty} \phi(p,t) \equiv 0. \tag{12}$$

The conditions (12) just determine the possibility to obtain the interpolation spline of desirable shape, intermediate between linear and cubic, in assumption of monotonicity via parameter p: $\phi(p,t) \le \phi(\bar{p},t)$ if $\bar{p} \ge p \ge 0$.

Note that various known generalizations of cubic splines besides hyperbolic splines are suited to described construction (8), for example,

– exponential spline [8, p.100]

$$\phi(p,t) = \frac{t^3 e^{-p(1-t)} - t}{p^2 + 6p + 6};$$

– rational spline due to Späth [9, 8]

$$\phi(p,t) = \frac{1}{2p^2 + 6p + 6}\left(\frac{t^3}{1 + p(1-t)} - t\right);$$

– rational spline due to Gregory [10, 11]

$$\phi(p,t) = \frac{1}{2p^2 + 8p + 6}\left(\frac{t^3}{1 + pt(1-t)} - t\right);$$

– spline with additional knots [12]

$$\phi(p,t) = \frac{(t - p(1-t))_+^3 - t}{6(p+1)^2};$$

– variable power spline (VP-spline) [13, 14]

$$\phi(p,t) = \frac{t^{3+p} - t}{p^2 + 5p + 6}.$$

The formula (8) and the constraints (11) ensure continuity of both $S(x)$ and $S''(x)$ on $[a,b]$. Then the requirement of continuity of the derivative $S'(x_i + 0) = S'(x_i - 0)$ implies

$$-\phi'(q_{i-1},0)\mu_i M_{i-1} + [\phi'(p_i,1)\mu_i + \phi'(q_i,1)\lambda_i]M_i - \phi'(p_{i+1},0)\lambda_i M_{i+1} = \delta_i, \tag{13}$$

$i = 1,\ldots,n-1$, where $\lambda_i = h_i/(h_{i-1} + h_i)$, $\mu_i = 1 - \lambda_i$. It is necessary to add two equations following from the boundary conditions (4) to complete the system respect to the unknowns $\{M_i\}$; these equations are

$$\phi'(q_0,1)M_0 - \phi'(p_1,0)M_1 = \delta_0, \qquad -\phi'(q_{n-1},0)M_{n-1} + \phi'(p_n,1)M_n = \delta_n. \tag{14}$$

Rewrite the system of equations (13), (14) in the form

$$\begin{cases} \phi'(q_0,1)h_0M_0 - \phi'(p_1,0)h_0M_1 = h_0\delta_0\,, \\ -\phi'(q_{i-1},0)h_{i-1}M_{i-1} + [\phi'(p_i,1)h_{i-1} + \phi'(q_i,1)h_i]M_i - \phi'(p_{i+1},0)h_iM_{i+1} \\ \qquad\qquad\qquad\qquad = (h_{i-1} + h_i)\delta_i, \qquad i = 1,\ldots,n-1, \\ -\phi'(q_{n-1},0)h_{n-1}M_{n-1} + \phi'(p_n,1)h_{n-1}M_n = h_{n-1}\delta_n\,. \end{cases} \tag{15}$$

3 Convexity Conditions for the Tension Splines

For the convexity of the spline of form (8), the nonnegativity of the solution of the system of equations (15) is necessary. However, nonnegative values of the second derivative at the points of mesh M_i does not guarantee its nonnegativity between knots, while the second derivative of an ordinary classic cubic spline is a piecewise linear function. It is clear from the form of the second derivative of generalized spline (10) that for ensuring the spline convexity it is necessary to set the constraint

$$\phi''(p,t) \geq 0, \quad t \in [0,1], \qquad \text{where} \;\; p \geq 0. \tag{16}$$

It is known [15, 16, 17] that the diagonal dominance in the system of linear equations makes it possible to write out sufficient conditions for the nonnegativity of its solution. The system (15) has diagonal dominance in columns if conditions

$$\phi'(p,0) < 0, \quad \phi'(p,0) + \phi'(p,1) > 0 \qquad \text{with } p \geq 0. \tag{17}$$

hold.

Theorem 3.1 ([5]). *Let the generalized cubic spline* (8) *interpolate convex data, and let relations* (11), (16) *and* (17) *be fulfilled. Then the spline* $S(x)$ *is convex if*

$$\begin{cases} \delta_0 + \dfrac{\delta_1\phi'(p_1,0)}{\mu_1\phi'(p_1,1)+\lambda_1\phi'(q_1,1)} \geq 0, \\ \delta_i + \dfrac{\delta_{i-1}\mu_i\phi'(q_{i-1},0)}{\mu_{i-1}\phi'(p_{i-1},1)+\lambda_{i-1}\phi'(q_{i-1},1)} + \dfrac{\delta_{i+1}\lambda_i\phi'(p_{i+1},0)}{\mu_{i+1}\phi'(p_{i+1},1)+\lambda_{i+1}\phi'(q_{i+1},1)} \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1,\ldots,n-1, \\ \delta_n + \dfrac{\delta_{n-1}\phi'(q_{n-1},0)}{\mu_{n-1}\phi'(p_{n-1},1)+\lambda_{n-1}\phi'(q_{n-1},1)} \geq 0. \end{cases} \tag{18}$$

There are the well-known sufficient conditions [15, 16] for the convexity of classic cubic splines

$$\delta_0 - \delta_1/2 \geq 0, \tag{19}$$

$$\delta_i - \delta_{i-1}\mu_i/2 - \delta_{i+1}\lambda_i/2 \geq 0, \quad i = 1,\ldots,n-1, \tag{20}$$

$$\delta_n - \delta_{n-1}/2 \geq 0. \tag{21}$$

Clearly, if the inequalities (19)–(21) are satisfied then conventional cubic spline is suitable for solving a convex interpolation problem, i.e. all tension parameters can be chosen zero and the inequalities (18) will be coincide with the inequalities (19)–(21). And so only in violation of any of these inequalities we have to introduce the functions $\phi(p,t)$ on correspondent intervals. Naturally, it is desirable to choose the design parameters which are minimal among those that ensure the inequalities (18).

If, for example, the ratio of neighboring δ_i less or equal to 2 then evidently the inequalities hold. Violation of any inequality in (19)–(21) indicates the fact that some value δ_i is significantly less than one or both its nearest-neighbors δ_{i-1} and δ_{i+1}. Application of the tension spline just should correct the situation. The coefficients for the corresponding values δ_{i-1} or δ_{i+1} in the i-th inequality of (18) are intended to decrease the influence of the corresponding terms. We formulate a general scheme of a choice of control parameters p_i and q_i and then, we describe algorithms for some special functions $\phi(p,t)$.

At the first step we check the conditions (19)–(21). If the conditions are fulfilled then the problem can be solved by the classical cubic spline. To construct a tension cubic spline, we must set the control parameters $p_1,\ldots,p_n$ and $q_0,\ldots,q_{n-1}$. The parameter p_i determines the tension to the left of the knot x_i, and parameter q_i determines the tension to the right of it. If conditions (19)–(21) are violated, we proceed to the second step. So we identify those "bad" knots in whose neighborhood it is required to increase a tension. Next, we denote two sets of their indices by P for parameters p_i and Q for parameters q_i. As already noted, violation of any of these inequalities for some i can be caused by a significant excess of δ_{i-1} and/or δ_{i+1} over δ_i therefore a number $i-1$ is added to the set Q and/or a number $i+1$ is added to the set P.

4 Determining the Tension Parameters

Let's give here the algorithm for determining the tension parameters proposed in [5].

1. Check the conditions (19)–(21). If all inequalities are fulfilled then we set all tension parameters equal to zeros and the problem can be solved by the classical cubic spline.

2. If some of these conditions are violated, we define two sets of knot numbers P and Q with nonzero tension. If inequality (19) is wrong, then we put the index 1 in the set P. Similarly, if inequality (21) is violated, then we put the index $n-1$ in the set Q. We look through the values i from 1 to $n-1$. If the inequality (20) does not hold, then at least one of inequalities

$$\delta_i - \delta_{i-1}/2 \geq 0, \tag{22}$$
$$\delta_i - \delta_{i+1}/2 \geq 0, \tag{23}$$

or both will not hold. We add the number $i-1$ in the set Q if inequality (22) is not satisfied, and/or the number $i+1$ in the set P if the inequality (23) is violated.

3. The set P consists of indices of knots in which it is required to set nonzero parameters p_i and the set Q consists of indices of knots in which it is required to set nonzero parameters q_i. To all other parameters we assign the value 0. For all indices $i \in P$, using formulas

$$\xi_i = \begin{cases} \left(\delta_{i-1} - \delta_{i-2}\mu_{i-1}/2\right)/\delta_i\lambda_{i-1}, & \text{for } i-2 \notin P, \\ \delta_{i-1}/\delta_i, & \text{for } i-2 \in P \cup Q \text{ or } i=1, \end{cases} \tag{24}$$

we calculate the quantities ξ_i and the quantities η_i by formulas

$$\eta_i = \begin{cases} \left(\delta_{i+1} - \delta_{i+2}\lambda_{i+1}/2\right)/\delta_i\mu_{i+1}, & \text{for } i+2 \notin Q, \\ \delta_{i+1}/\delta_i, & \text{for } i+2 \in P \cup Q \text{ or } i=n-1, \end{cases} \tag{25}$$

for all indices $i \in Q$.

4. If $0 \in Q$ and/or $n \in P$, then we solve the equations

$$-\phi'(q_0,0)/\phi'(q_0,1) = \eta_0, \qquad -\phi'(p_n,0)/\phi'(p_n,1) = \xi_n \tag{26}$$

with respect to q_0 and/or p_n. We look through the values i from 1 to $n-1$.

4.1. Case $i \in P$ and $i \notin Q$. We solve the equation

$$\frac{-\phi'(p_i,0)}{\mu_i\phi'(p_i,1) + \lambda_i/3} = \xi_i \tag{27}$$

with respect to p_i and verify the inequality

$$\delta_{i+1} + \frac{\delta_i\mu_{i+1}\phi'(q_i,0)}{\mu_i\phi'(p_i,1) + \lambda_i\phi'(q_i,1)} + \frac{\delta_{i+2}\lambda_{i+1}\phi'(p_{i+2},0)}{\mu_{i+2}\phi'(p_{i+2},1) + \lambda_{i+2}\phi'(q_{i+2},1)} \geq 0$$

with $i+2 \notin Q$, $i < n-1$ or

$$\delta_{i+1} + \frac{\delta_i \phi'(q_i, 0)}{\mu_i \phi'(p_i, 1) + \lambda_i \phi'(q_i, 1)} \geq 0 \tag{28}$$

with $i = n-1$ or $i+2 \in Q$ for this p_i and $q_i = 0$. In the case of violation we set $i \in Q$ and we calculate the quantities η_i by formulas (25). Go to 4.3.

4.2. Case $i \notin P$ and $i \in Q$. We solve the equation

$$\frac{-\phi'(q_i, 0)}{\mu_i/3 + \lambda_i \phi'(q_i, 1)} = \eta_i \tag{29}$$

with respect to q_i and verify the inequality

$$\delta_{i-1} + \frac{\delta_{i-2}\mu_{i-1}\phi'(q_{i-2}, 0)}{\mu_{i-2}\phi'(p_{i-2}, 1) + \lambda_{i-2}\phi'(q_{i-2}, 1)} + \frac{\delta_i \lambda_{i-1} \phi'(p_i, 0)}{\mu_i \phi'(p_i, 1) + \lambda_i \phi'(q_i, 1)} \geq 0$$

with $i-2 \notin P$, $i > 1$ or

$$\delta_{i-1} + \frac{\delta_i \phi'(p_i, 0)}{\mu_i \phi'(p_i, 1) + \lambda_i \phi'(q_i, 1)} \geq 0 \tag{30}$$

with $i = 1$ or $i-2 \in P$ for $p_i = 0$ and this q_i. In the case of violation we set $i \in P$ and we calculate the quantities ξ_i by formulas (24). Go to 4.3.

4.3. Case $i \in P$ and $i \in Q$. We solve the system of equations

$$\begin{cases} \xi_i/\phi'(p_i, 0) - \eta_i/\phi'(q_i, 0) = 0, \\ \xi_i \mu_i \phi'(p_i, 1)/\phi'(p_i, 0) + \eta_i \lambda_i \phi'(q_i, 1)/\phi'(q_i, 0) + 1 = 0 \end{cases} \tag{31}$$

with respect to p_i and q_i.

5 Piecewise Convex Interpolation by Tension Splines

We propose to adapt the algorithm described above for data interpolation if it is necessary to preserve convexity in subintervals where the data are convex.

Usually, local splines are used to solve such a problem. However, in most problems related to data interpolation, the main and most universal tool is still the classical cubic spline of class C^2. The tension splines we consider are a generalization of the classical cubic spline. Moreover, our algorithm works in such a way that if for some specific data the classical cubic interpolation spline gives acceptable results, our algorithm gives zero control parameters, i.e. the tension spline becomes a conventional cubic spline. Note that we can correctly apply our algorithm only for fully convex data.

The tension splines considered here have smoothness C^2 and are not local. Generally speaking, such tension splines avoid undesirable oscillations if the control parameters are chosen correctly, because the spline approaches piecewise linear interpolation when the control parameters are significantly increased. It is shown that increasing several parameters has a significant effect locally. For example, an increase in parameter p_i causes the spline link to tension to the left of knot x_i, and parameter q_i to the right of x_i in the corresponding mesh interval adjacent to the knot. As we move away from knot x_i, the influence of parameters p_i and q_i weakens and is almost imperceptible outside these intervals. The latter fact gives us the opportunity to choose the control parameters not over the entire data domain, but locally, on the subdomain of the data where the direction of convexity does not change.

Since our algorithm does not allow us to directly and automatically obtain the values of the control parameters that ensure the fulfilment of the desired conditions for different signs of convexity, we propose to divide the original data into sections with the same signs of convexity, and any three points of the section should have the same sign of the second divided difference.

It is clear that it makes sense to talk about preserving the sign of convexity by an interpolation function if such sections consist of more than three points. Thus, dividing the initial data into sections of the same convexity, we have several sections with positive or negative convexity, but it may also turn out that some of the source data will not be included in the resulting sections (sections of only three points).

Now the task is as follows. For each convexity segment of the same sign, separately construct a tension spline using our algorithm (see Section 4). Thus, for each data segment we obtain a set of tension parameters. We can now proceed to construct a global tension spline over the entire set of original data. At each mesh interval, we take as control parameters exactly those parameters that were obtained by our algorithm when interpolating in regions with the same convexity. There may be areas where we did not build a tension spline locally, where we do not have calculated tension parameters. But these are areas where the signs of second divided differences are alternating, and it makes no sense to talk about any inheritance.

Therefore, we propose to set the control parameters in these regions equal to zero. In fact, in a particular practical problem, the parameters in these regions can be adjusted for some additional reasons.

Example 1. The referee of [5] suggested us to consider the famous Akima data [18]. The Akima data set consists of 11 points, but the data are not convex. We will only consider the first 9 points ($n = 8$) since this data is convex. The first 6 points lie on the same straight line and hence the second divided differences $\delta_1, \dots, \delta_4$ are equal to 0. Our algorithm is applicable only for strictly convex data. To make the data strictly convex, we slightly modify the function values for only the first 6 points so that these modified points lie on a parabola. This modification is shown in Tab. 1.

Table 1: Modified Akima data.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	10	10.0004	10.0016	10.0036	10.0064	10.01	10.5	15	50	60	85

To construct a classical cubic spline or generalized spline, we need two end conditions — the derivatives of $f(x)$ at the ends. We define the derivatives as the values of the derivative of the parabola passing through the 3 points at the ends.

The classical cubic spline C^2 for the modified original full Akima data is shown in Fig. 1 (dashed line). Note that the changes made to the Akima data are so slight that they do not visually change the spline. The curves in Fig. 1 have unwanted waves. We apply our algorithm to the shortened Akima data. We find violations of inequalities (20) at $i = 4, 5, 6$. Then we find the sets $P = \{5, 6, 7\}$, $Q = \emptyset$ and $\xi_5 = 0.002$, $\xi_6 = 0.242$, $\xi_7 = 0.248$.

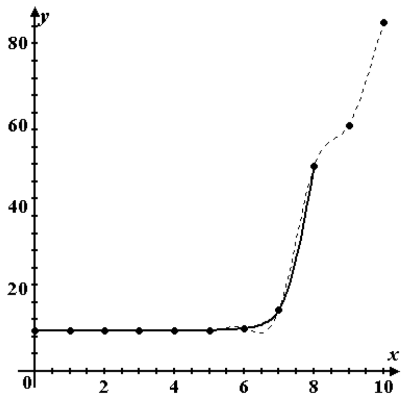

Figure 1: The cubic spline for complete Akima data (dashed line) and the Späth rational spline for modified trimmed data.

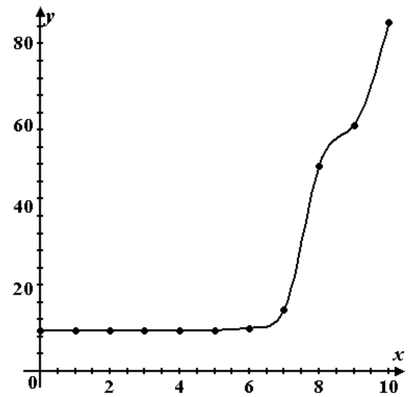

Figure 2: The Späth rational spline for complete Akima data.

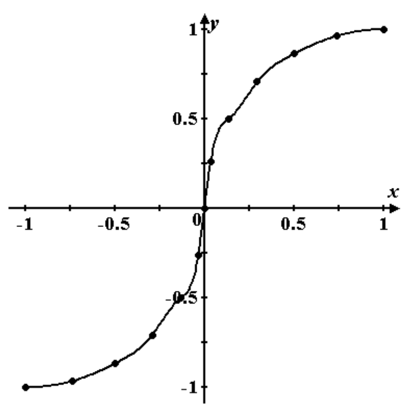

Figure 3: The cubic spline for data of two quarter circles.

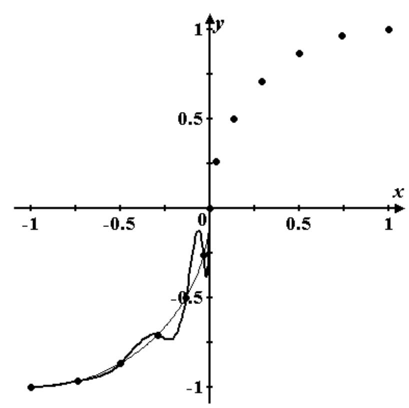

Figure 4: The cubic spline from quarter circle data.

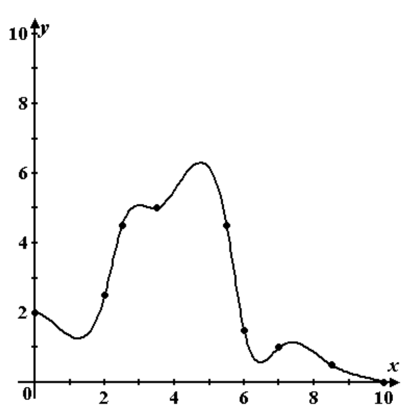

Figure 5: The cubic spline for Späth data.

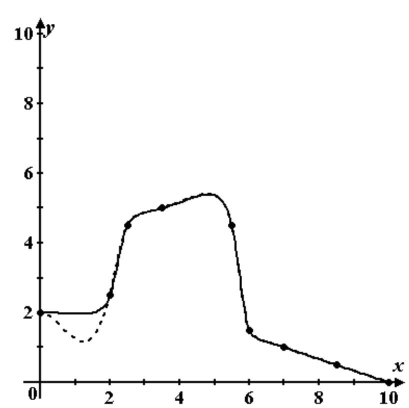

Figure 6: The Späth rational spline for Späth data.

For the rational Späth spline, we find the parameters $p_5 = 32.6$, $p_6 = 1.13$, $p_7 = 1.09$ from equation (27). For these parameters, the inequalities (28) are fulfilled. The bold line in Fig. 1 shows the convex tension spline graph for the modified shortened Akima data.

The rational Späth spline for the full Akima data with the same tension parameters is also convex on the interval [0,8] (see Fig. 2).

For other types of tension splines, the results and graphs are similar.

Example 2. The second data set consists of points uniformly spaced at 15 intervals on two quarters of a circle of radius equal to one, with zero derivatives as end conditions. This example is taken from the paper [19]. The classical cubic spline interpolating the data on a quarter circle is not convex, the spline has inflection points at each interval of the partition (Fig. 3).

The complete data set can be divided into two groups of 7 points each (there are 13 points in total) with the same convexity direction. The first group of data consists of points numbered 0 to 6, here the second divided differences are positive. The second group contains points 6 to 12, here the second divided differences are negative.

We will construct two separate tension splines, preserving the sign of the convexity of the data. Since the example is symmetric, we will limit our consideration to the interval $[x_0, x_6] = [-1, 0]$. The condition for the right end of the derivative will be $+50$, replacing the infinite gradient (see [11, 20]). The classical cubic spline is shown in Fig. 4. Let us check our sufficient conditions for convexity in the first region $[-1, 0]$. There are violations of inequalities (20) at $i = 4, 5$.

Applying our algorithm for selecting tension parameters for generalized cubic splines, we find the sets $P = \{5, 6\}$, $Q = \emptyset$.

For $5 \in P$ we have only one tension parameter p_5,which is determined from the first equation (26) at $\xi_5 = 0.257$. For $6 \in P$, we determine the parameters p_6 from equation (29) for the value $\xi_6 = 0.117$. For the rational Späth spline we have $p_5 = 1.28$, $p_6 = 6.54$.

The rational Späth spline for the full data has these nonzero tension parameters: $p_5 = 1.28$, $p_6 = 6.54$, $q_6 = 6.54$, $q_7 = 1.28$.

A similar situation occurs with other types of generalized splines.

Example 3. These data are often used as examples in Späth's monographs [8, 4], and are summarised in Tab. 2. To construct the C^2 spline, we need to add boundary conditions. At the left end we assume that $f'(0) = 0$, and at the right end it is natural to assume that the derivative is equal to the divided difference from the data at that end.

Table 2: Späth data.

i	0	1	2	3	4	5	6	7	8
x_i	0	2	2.5	3.5	5.5	6	7	8.5	10
f_i	2	2.5	4.5	5	4.5	1.5	1	0.5	0

Conventional cubic interpolation spline for the Späth data shown in Fig. 5. If we apply our reasoning from the beginning of the section, the full data set of 9 points

($n = 8$) can be divided into 3 subsets with the following point numbers: $J_1 = \{0, 1, 2\}$, $J_2 = \{1, 2, 3, 4, 5\}$ and $J_3 = \{4, 5, 6, 7, 8\}$. The first set contains only 3 points and the others contain 5 points each, for each set the task of constructing interpolation tension splines that ensure inheritance of the convexity of the corresponding data sets is set.

Having constructed a tension spline on the set J_2, we see that inequality (20) is not satisfied at $i = 3$, so $2 \in Q$ and $4 \in P$. We find $\eta_2 = 0.107$, $\xi_4 = 0.109$. For the rational Späth spline, we find the parameters $q_2 = 3.48$ and $p_4 = 4.12$ from equations (29) and (27), respectively. We then set the other control parameters to zero. With these control parameters, all sufficient conditions are fulfilled and the tension spline will be convex.

Checking inequality (20) for the interpolation data from the third set J_3 shows a violation for $i = 6, 7$, so that $5, 6 \in Q$. Here, as in Example 1, we have slightly changed the value of f_8 so that the last points do not lie on the same straight line. From equations (27) we find $\eta_5 = 0.045$, $\eta_6 = 0.008$ and the control parameters $q_5 = 6.95$, $q_6 = 19.58$ for the rational Späth spline. But it turns out that with these parameters $p_5 = 0$ and $q_5 = 6.95$, the inequality (30) is not fulfilled, which means that the parameter p_5 must be nonzero, and we proceed to point 4.3 of the parameter determination algorithm. In this case, the parameters p_5 and q_5 must be found from the system of equations (31). We obtain $p_5 = 0.16$, $q_5 = 7.25$. Now all necessary conditions are fulfilled and the tension spline for the data set should be convex.

Let's construct the interpolation tension spline over the full data set, taking the control parameters of the splines on J_2 set and on J_3 set. The resulting spline is shown in Fig. 6 (dashed line).

In this example, it seems appropriate for us to achieve convexity on the set J_1, consisting of only three data points. Checking the inequalities (19)–(21) when constructing an interpolating cubic spline reveals the failure of the inequality (19), $1 \in P$. We calculate $\xi_1 = 0.083$. If we assume $q_1 = 0$, then from the equation (27) we find $p_1 = 5.26$, but in this case the inequality (28) does not hold, i.e. parameter q_1 must not be zero. Then p_1 and q_1 are found from the system of equations (31) — $p_1 = 5.74$, $q_1 = 0.42$.

If in the final tension spline we additionally set the parameters p_1 and q_1 with the found values, then the shape of the interpolant will be corrected at the first mesh interval. This spline is shown in Fig. 6 (thick line).

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