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STABLE ITERATIVE METHODS IN PROBLEM OF CONSTRAINED CONVEX MINIMIZATIONS

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Abstract We investigate the ill-posed problem of minimizing a convex functional in a Hilbert space. Previously, in the author's work, the problem of minimizing a quadratic functional was considered. The iterative processes that are stable to disturbances of all input data were constructed and studied. In this paper the developed approach is generalized to the constrained convex minimization. The proposed methods generate regularizing algorithms for quadratic and convex minimization problems, in contrast to direct methods that do not have this property.

Keywords: ill-posed problem, linear and convex constraints, iterative process, convex minimization, regularizing algorithm.

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1 Problem statement and preliminary information

We study the problem of a constrained convex minimization

$$\inf\{f_0(u) : u \in Q\} = F, \tag{1.1}$$

where f_0 is a convex continuous functional, Q is a convex closed subset of the Hilbert space U. Along with problem (1.1) the main object of our research is the classical convex programming problem, for which the set Q is specified by a system of convex inequalities, i.e.

$$Q = \{u \in U : f_i(u) \le 0, \ i = 1, 2, ..., m\}, \tag{1.2}$$

where f_i are convex continuous functionals.

Problem (1.1), (1.2) are among the ill-posed problems in which there may be no continuous dependence of the solution on the input data of the problem. Therefore, under the conditions of noisy data $\{\tilde{f}_i\}$, direct methods [1] for constructing a stable family of approximate solutions are not applicable. Corresponding examples of conditional convex minimization are presented in [2]. It is also outlined there a method for reducing problem (1.1), (1.2) to an ill-posed problem of non-constraint convex minimization based on Tikhonov regularization and the method of penalty functions. A similar approach to reducing an ill-posed linear programming problem was studied in [3]. When solving the problem (1.1) by functional, along with the method of penalty functions, the method of barrier functions is used in conjunction with the method of quasi-solutions as a regularization algorithm [4].

A particular version of the problem (1.1) is the problem of minimizing the quadratic functional

$$\inf\{\|Au - f\|^2 : u \in Q\}, \tag{1.3}$$

where A is a linear bounded operator acting on a pair of Hilbert spaces U, F. For a set of constraints Q, specified by systems of linear equalities and inequalities, in the work [1] direct methods for solving the problem (1.3) are presented. Note that in the general situation these methods are unstable to input data errors. It should be noted that, on the one hand, the problem (1.1), (1.3) is an object of study of mathematical programming, which has numerous applications in various disciplines, and on the other hand, the problem (1.3) can be considered as a generalization of the formulation of the inverse ill-posed problems in the form of an operator equation with a priori information [5]

$$Au = f,\ u \in Q,\ M \cap Q \neq \varnothing,$$

where M is the set of solutions to the operator equation. In the case when the operator equation does not have a usual solution, we are forced to move on to the minimization problem in the form 1.3. This class of ill-posed problems is found in all branches of natural science. In the work [6] for a set of restrictions $Q = \{u \in U : Bu = b,\ Du \leq d\}$ an iterative method was announced that is stable to errors in all input data, in contrast to methods of the work [1], which do not have this property.

Definition 1. Problem 1.1 is called well-posed according to Hadamard if: 1) a solution exists; 2) there is only one solution; 3) the solution continuously depends on f_0 and Q (in each specific situation the concept of continuity will be clarified).

Definition 2. Problem (1.1) is called well-posed according to Tikhonov if: 1) a solution exists; 2) there is only one solution; 3) each minimizing sequence $u_n \in Q$, $f_0(u_n) \to F$ converges to a unique solution $\hat{u}$ to problem (1.1).

Definition 3. If at least one of conditions 1-3 in Definitions 1, 2 is violated, then they say that the problem is formulated ill-posed according to Hadamard (Tikhonov).
In the future, as a rule, we consider problems to be ill-posed in which condition 3 in Definitions 1 and 2 is violated.

Definition 4. The sequence $Q_n \in U$ converges to the set $Q \in U$ in the Hausdorff sense if

$$\lim_{n\to\infty} \alpha(Q_n, Q) = \lim_{n\to\infty} \max\{\beta(Q_n, Q), \beta(Q, Q_n)\} = 0,$$

where $\beta(Q_n, Q) = \sup_{u\in Q_n}, \inf_{v\in Q} \|u - v\|$.

Definition 5. A sequence of sets Q_n converges to a set Q in the Mosco sense if :
a) $\forall u \in Q\, \exists \{u_n\} : u_n \in Q_n, u_n \to u$;
b) $\forall \{u_{n_j}\} \in Q_{n_j}, u_{n_j} \rightharpoonup u \Rightarrow u \in Q$, where the symbol " $\rightharpoonup$ " means weak convergence.

Under the assumption that problem (1.1) has a unique solution, in the work [7] two statements are formulated and proven that clarify the features of the introduced definitions of well-posedness in terms of the convergence of admissible sets in the sense of Hausdorff and Mosco and their relationship is established.

Statement 1 [7]. If the minimization problem 1.1 on convex closed sets Q is well-posed according to Hadamard with respect to the Hausdorff convergence $Q_n \to Q$, then on every closed convex it is well-posed according to Tikhonov.

Statement 2 [7]. Let the functional f_0 be additionally uniformly continuous on each bounded set. Then well-posednes in the Tikhonov sense on every closed convex set Q implies Hadamard well-posedness with respect to Mosco convergence $Q_n \to Q$.

2 Iterative method of convex minimization on a set of linear constraints

As is well known, (see, for example, Theorems 21.1, 21.2 in [5]) problem 1.1 is equivalent to the operator equation

$$u = P_Q(u - \lambda \nabla f_0(u)) \equiv T(u), \tag{2.1}$$

those the problem of finding a fixed point of the operator T. Here the set Q is defined by a system of convex sets 1.2, P_Q is a metric projection onto the set Q, λ is a positive parameter. As a method for constructing a set of approximate solutions, we will use the method of successive approximations with the step operator T, modified using correction factors γ_k [9]

$$u^{k+1} = (1 - \gamma_{k+1})T(u^k) + (1 - \gamma_{k+1})v_0, \tag{2.2}$$

where the operator T is defined by the formula 2.1, v_0 is an arbitrary element from U, with the help of which, if necessary, one can take into account a priori (physical) information about the solution in case of its non-uniqueness. Let us study the convergence of process 2.2 with exact data.

Theorem 1. Let problem 1.1 be solvable and let $\hat{u}$ be its v_0-normal solution. Let

$$\sup\{\|f''(u)\| : \ u \in Q\} \leq N, \ \ \lambda < 2/N \tag{2.3}$$

and the sequence γ_k is admissible (see definition in [9]). Then for any u^0 the sequence $\{u^k\}$ constructed by the process 2.2 converges to the v_0-normal solution $\hat{v}$ of problem (1.1).

Proof: According to the work of [8] (see Lemma 4.1), if the condition 2.3 is satisfied, the operator is nonexpansive (we denote this class by $\mathcal{K}$). In addition, the operator T operator maps the ball to itself: $\|u-\hat{u}\| \leq r \Rightarrow \|T(u)-T(\hat{u})\| \leq \|u-\hat{u}\| \leq r$. According to the theorem proven in [9], this implies strong convergence of iterations in the Hilbert space U

$$\lim_{k\to\infty} \|u^k - \hat{u}\| = 0, \, \hat{u} \in Fix(T),$$

where $\hat{u}$ is v_0-normal solution of problem 1.1, in particular, of problem 1.1, (1.2)

Let's study the convergence of the process 2.2 with approximate data

$$\tilde{u}^{k+1} = (1 - \gamma_{k+1})[P_{\tilde{Q}}(\tilde{u}^k - \nabla \tilde{f}_0(\tilde{u}^k))] + (1 - \gamma_{k+1})v_0, \tag{2.4}$$

where $\tilde{u}^0 = u^0$, the set $\tilde{Q}$ is defined by the system 1.2, in which $f_i(u)$ are replaced by noisy functions $\tilde{f}_i(u)$. Let us introduce the notation: $F(u) = u - \lambda \nabla f_0(u)$, $\tilde{F}(u) = u - \lambda \nabla \tilde{f}_0(u)$.

Theorem 2. Let us assume that problem 1.1 is solvable, $\hat{u}-$ is its v_0-normal solution, and the following conditions are satisfied:

$$\sup\{\|f_0''(u^k)\|, \, \|\tilde{f}_0''(u^k)\|\} \leq N, \, \lambda < 2/N; \tag{2.5}$$

$$\|\nabla f_0(u^k) - \nabla \tilde{f}_0(u^k)\| \leq c_1\delta; \tag{2.6}$$

$$\|P_Q\tilde{F}(u^k) - P_{\tilde{Q}}\tilde{F}(u^k)\| \leq c_2\delta. \tag{2.7}$$

Then, if choosing the number of iterations in accordance with the rule $k(\delta)\delta \to 0, \ \delta \to 0$, convergence holds

$$\lim_{\delta\to 0} \|\tilde{u}^{k(\delta)} - \hat{u}\| = 0.$$

Proof: From the triangle inequality for norms we have

$$\|\hat{u} - \tilde{u}^{k+1}\| \leq \|\hat{u} - u^{k+1}\| + \|u^{k+1} - \tilde{u}^{k+1}\|, \tag{2.8}$$

where the first term on the right side of the inequality decreases to zero as $k \to \infty$ according to theorem 1.

Taking into account the condition that the condition 2.5 implies belonging of the operators $T = P_Q F$, $\tilde{T} = P_{\tilde{Q}} \tilde{F}$ ([8], Lemma 4.1) class $\mathcal{K}$ of nonexpanding operators and the inequalities 2.6, 2.7 are satisfied, we obtain an estimate for the second term in 2.8

$$\begin{aligned}
\|u^{k+1} - \tilde{u}^{k+1}\| &\le \|P_Q(u^k) - \lambda\nabla f_0(u^k)) - P_Q(u^k - \lambda\nabla\tilde{f}_0(u^k))\| \\
&+ \|P_Q(u^k - \lambda\nabla\tilde{f}_0(u^k)) - P_{\tilde{Q}}(u^k - \lambda\nabla\tilde{f}_0(u^k))\| \\
&+ \|P_{\tilde{Q}}(u^k - \lambda\nabla\tilde{f}_0(u^k)) - P_{\tilde{Q}}(\tilde{u}^k - \nabla\tilde{f}_0(\tilde{u}^k))\| \\
&\le \lambda c_1\delta + c_2\delta + \|u^k - \tilde{u}^k\| \le 2(\lambda c_1 + c_2)\delta + \|u^{k-1} - \tilde{u}^{k-1}\| \\
&\le \ldots \le (\lambda c_1 + c_2)\delta \cdot (k+1),
\end{aligned}$$

which together with Theorem 1, completes the proof of Theorem 2.

Example 2.1. Let us consider the problem of finding a normal solution, i.e. metric projection of zero, for the simplest system of inequalities

$$\begin{aligned}
&\min\{\|z\|^2 : 0 \le z_1 \le 1,\ z_1 - z_2 \le 0,\ -z_1 + z_2 \le 0\} = F, \\
&F_{opt} = 0,\ \ z_{opt} = (0,\ 0)^{\mathrm{T}}; \\
&\min\{\|z\|^2 : 0 \le z_1 \le 1,\ z_1 - (1+\epsilon)z_2 + \epsilon \le 0, -z_1 + (1-\epsilon)z_2 + \epsilon \le 0\},\ \epsilon \ge 0, \\
&F^{\epsilon}_{opt} = 2,\ \ z^{\epsilon}_{opt} = (1,\ 1)^{\mathrm{T}}.
\end{aligned}$$

Example 2.1 shows that finding the projection onto a convex closed set Q is, in the general case, an unstable problem to perturbations of the admissible set Q. Therefore, in this situation, the estimate 2.7 may not hold; hence, we will not be able to use the general estimate obtained in Theorem 2 to justify the convergence of the basic method. To form a regularizing algorithm for the problem 1.1, 1.2 based on the process 2.4 it is necessary to construct a stable method for calculating the projection onto Q. Note that such a regularizing algorithm for calculating the projection was constructed and announced in the work [6] for the set Q, specified in the form of systems of linear equalities and inequalities in the study of the quadratic minimization problem. Let us show that this is a stable algorithm for finding the projection for Q in the form of linear constraints under more general conditions to justify the convergence of the process 2.4, replacing the condition 2.7 by this algorithm calculating with accuracy δ the projection P_Q at every step process 2.4.

Let us briefly outline the iterative method, which generates a stable algorithm for calculating the metric projection in the case of the set Q, defined by linear constraints

$$Q = \{u : l_i(u) = 0, i \in J_1,\ l_i(u) \le 0,\ i \in J_2\} \ne \varnothing. \tag{2.9}$$

Let us form an operator

$$\overline{T}(u) = u - (1/\kappa)[\sum_{J_1} l_i(u)a_i + \sum_{J_2} l_i(u)a_i], \tag{2.10}$$

where, $\kappa = \sum_{J_1 \bigcup J_2} \|a_i\|^2$, which is a convex sum of projections, either a projection onto a hyperplane or onto a half-space and Fix(T)=Q, and create the iterative process

$$u^{k+1} = \gamma_{k+1}\overline{T}(u^k) + (1 - \gamma_{k+1})v_0, \tag{2.11}$$

where γ_k is an admissible sequence, for example, $\gamma = 1 - k^{-p}$, $0 < p < 1$. Since $\overline{T}$ is a nonexpanding operator, then, using the argumentation in the proof of Theorem 1, we conclude

that the iterative sequence $\{u^k\}$, generated by process (2.11) converges to the projection $\bar{v} = P_Q$ of element v_0 onto set Q.
Lemma 1. Let the set Q defined by the formula 2.9, may be empty. Let

$$\tilde{M} = \operatorname{Arg}\min\{\nabla(\sum_{J_i} l_l^2(u)a_i) + \sum_{J_2} l_i^{+2}(u)a_i)\} \neq \varnothing$$

Then the operator

$$\tilde{T}(u) = (u - \lambda/\kappa)\nabla(\sum_{J_1} l_i^2(u)a_i + \sum_{J_2} i_l^2(u)a_i)$$

satisfies relations $\mathrm{Fix}(\tilde{T}) = \tilde{M},\ \tilde{T} \in \mathcal{K}$.
Proof: The first relation follows from the equivalence of the minimization problem and the problem of finding a fixed point (see (2.1)) , the equality $\tilde{T} = \overline{T}$ and definition of the operator $\tilde{T}$, the second relation is a consequence of the fact that the operator $\tilde{T} = \overline{T}$ is a convex sum of projections; here $\overline{T}$ is defined by (2.10).
Corollary 1. If $Q \neq \varnothing$, then process

$$\tilde{u}^{k+1} = u^k - \gamma_{k+1}\tilde{T}(\tilde{u}^k) + (1 - \gamma_{k+1})v_o \tag{2.12}$$

converges to the element $\hat{u} \in Q$ closest to v_0, i.e. projections v_0 onto the set Q. Moreover, if the vectors $\{a_i, b\}$ when specifying the set Q of linear constraints are specified with an error $\|\tilde{a}_i - a_i\| \leq \delta, \|\tilde{b} - b\| \leq \delta$, then if choosing the number of iterations in accordance with the rule $k(\delta)\delta \to 0$, convergence $\tilde{u}^{k(\delta)} \to \hat{u}$ as $\delta \to 0$ holds. i.e. method (2.12) generates a regularizing algorithm.

Combining Corollary 1 with Theorem 2, we obtain
Corollary 2. The iterative process (2.4) with the implementation of the metric projection P_Q by method (2.12) with an accuracy of δ generates an algorithm that is stable to input data errors.

Note that Lemma 1 and Corollary 1 are a generalization of Theorem 9 from the paper [6] to the case of inconsistent systems of linear constraints.

3 Iterative method of convex minimization in the general case

Let us consider in more detail the problem (1.1), when the set of constraints has the form (1.2) in the presence of errors in the form

$$|f_i - f_i^\delta| \leq \delta,\ i = 0, 1, ..., m. \tag{3.1}$$

As in the case of specifying a set of constraints Q in the form of systems of equalities and inequalities (2.9), to construct a regularizing algorithm for solving a convex programming problem based on the method (2.4), we need a stable method for calculating the metric projection onto a system of convex inequalities (1.2), i.e algorithm for solving the following problem:

$$\inf\{\|u - v_0\|^2 : u \in U,\ f_i(u) \leq 0,\ i = 1, 2, ..., m\}. \tag{3.2}$$

When constructing such an algorithm, we use the operator as the step operator of the basic iterative method

$$S(u) = \sum_{i=1}^{m} \alpha_i S_i(u), \alpha_i > 0, \sum_{i=1}^{m} \alpha_i = 1, \tag{3.3}$$

where

$$S_i(u) = u - \lambda(f_i^+(u)\nabla f_i(u))/\|\nabla f_i(u)\|^2, \tag{3.4}$$

if $\nabla f_i(u) \neq 0$, and $S_i(u) = u$ if $\nabla f(u) = 0$. When constructing such an algorithm, as a step operator of the basic iterative method, we use the operator proposed by Eremin (see, for example, [10], Ch.3, Sec.3) for finding a solution of a system of convex inequalities in R^n by the method of successive approximations $u^{k+1} = S(u^k)$.

Theorem 3. Let the gradients of the functionals f_i, $i = 1, 2, ..., m$ be bounded mappings, i.e. for any bounded B of Hilbert space U

$$\sup\{\|\nabla f_i(u)\|;\ u \in B\} \leq c < \infty.$$

Then the operator S, defined by the formulae (3.3), (3.4) is strongly Fejér and generates a sequence of iterations $u^{k+1} = S(u^k)$ for which the following properties hold:
1) $u^k \rightharpoonup \hat{u}$, $\hat{u} \in \text{Fix}(S) = Q$, where Q is defined in (1.2)
2) $\inf_z\{\lim \|u^k - z\| : z \in\text{Fix}(S)\ \}=\lim_{k\to\infty} \|u^k - \hat{u}\| = 0$.

Proof: By direct verification one can verify that the set of fixed points $\text{Fix}(S_i)$ coincides with the set $Q_i = \{u \in U : f_i(u) \leq 0\}$, therefore, $\text{Fix}(S) = Q$, where Q is given by the formula (1.2). If for $u \notin Q_i, z \in Q_i$, we use the well-known inequality $< \nabla f(u), z - u > \leq f(z) - f(u)$ in the equality

$$\|S_i(u) - z\|^2 = \|u = z\|^2 - 2\lambda \frac{f_i(u) < \nabla f_i(u), u - z >}{\|\nabla f_i(u)\|^2} + \lambda^2 \frac{f_i^2(u)}{\|\nabla f_i(u)\|^2},$$

then we obtain the following estimate:

$$\begin{aligned} \|S_i(u) - z\|^2 &\leq \|u - z\|^2 - 2\lambda \frac{f_i(u)(f_i(u) - f_i(z))}{\|\nabla f_i(u)\|^2} + \lambda^2 \frac{f_i^2(u)}{\|\nabla f_i(u)\|^2} \\ &\leq \|u - z\|^2 - \frac{2-\lambda}{\lambda}\|u - S_i(u)\|^2. \end{aligned} \tag{3.5}$$

This estimate 3.5 means that the operator S_i belongs to the class of strongly Fejér operators $\mathcal{P}_{Q_i}^{\nu_i}$ for $\nu_i = (2 - \lambda)/(\lambda)$, which implies ([5], Theorem 17.1) operator S belongs to class $\mathcal{P}_{Q_i}^{\nu}$ for $\nu = (2 - \lambda)/(\lambda)$.

Let us make sure that the following property holds for the operator S_i

$$S_i : u_k \rightharpoonup \bar{u}, u_k - S_i(u_k) \to 0 \Rightarrow \bar{u} \in \text{Fix(S}_\text{i}). \tag{3.6}$$

Indeed, let the premise be satisfied, then, due to the boundedness of the gradients of the functionals, we have $u_k \rightharpoonup \bar{u}$, $\ f_i^+(u^k)/\|\nabla f_i(u_k)\| \to 0$. Due to the weak lower semicontinuity of the convex continuous functional, we obtain

$$f_i(\bar{u}) \leq f_i^+(\bar{u}) \leq \liminf_{k\to\infty} f_i(u_k) = 0 \ \Rightarrow \bar{u} \in Q_i = \text{Fix}(S_i)\ \forall i,$$

Consequently, the property proved for S_i is valid for the operator S if the sequence $\{u_k\}$ is iterative ([11], Lemma 2.10).

Substituting the iteration point u^k into the inequality 3.5, we obtain the following inequality

$$\|u^{k+1} - z\|^2 \leq \|u^k - z\|^2 - \nu\|S(u^k) - u^k\|^2,$$

from which it follows for any $z \in\text{Fix}(S)$

$$\lim_{k\to\infty} \|u^k - z\| = d,\ \lim_{k\to\infty} \|S(u^k) - u^k)\| = 0. \tag{3.7}$$

From the first relation in 3.7 it follows the existence of a weakly convergent subsequence $u_i^k \rightharpoonup \hat{u}$, which, together with 3.6 and the second relation in 3.7 implies $\hat{u} \in \mathrm{Fix}(S)$. The uniqueness of the weakly limit point and property 1 are proved in a similar way. The proof of property 2 of the theorem follows from the representation

$$\|u^k = z\|^2 = \|u^k - \hat{u}\|^2 + 2 < u^k - \hat{u}, \hat{u} - z > + \|u^k - z\|^2,$$

in which we need to go to the limit as $k \to \infty$ and take into account that $u^k \rightharpoonup \hat{u}$.

Corollary 3. If the process $u^{k+1} = S(u^k)$ converges strongly, then the limit point $\hat{u}$ has the smallest norm among all $u \in$Fix(S).

Poof: follows from the inequality

$$\|\hat{u}\| \leq \lim_{k\to\infty} \inf\{\|\hat{u} - u^k\| + \|u^k - z\| + \|z\| : z\}.$$

and proven properties 2.

Corollary 4. Let $U = R^n$ and f_i be differentiable functionals. Then the iterative process converges to the solution of problem 3.2, i.e. metric projection v_0 onto the set Q, defined by a system of convex inequalities 1.2.

Let us define the set of constraints Q_n for problem (3.2) in the case of approximate data $f_i^{\delta_n}$ in the form

$$Q_n = \{u \in R^n : f_i^{\delta_n}(u) - \delta_n \leq 0, \, i = 1, 2, ..., m\}, \tag{3.8}$$

where $|f_i(u) - f_i^{\delta_n}(u)| \leq \delta_n$, $\delta_n \to 0$.

Lemma 2. Problem 3.2 is well-posed according to Hadamard with respect to Mosco convergence.

Proof: Indeed, the objective function f_0 is a uniformly continuous function on each bounded set of R^n. In addition, f_0 is a strongly convex function, which implies that the problem is well-posed according to Tikhonov (3.2) on every convex set, in particular, for Q, given by the formula 1.2. This is a consequence of the well-known inequality

$$\|u - \hat{u}\|^2 \leq (2/\kappa)(f_0(u) - f(\hat{u})),$$

where $\hat{u}$ is the solution to problem 3.2, κ is the constant from the definition of strong convexity. Due to the continuity and convexity of the functions $f_i, f_i^{\delta_n}$, the sets Q, Q_n are convex and closed. Let us verify that Q_n converges under Mosco to the set Q. Since $Q \subseteq Q_n$, condition a in Definition 6 is satisfied automatically. Let $u_{n_j} \in Q_{n_j}$ and $u_{n_j} \to \bar{u}$. We have the following relations:

$$f_i(\bar{u}) = \limsup_{j\to\infty} f_i(u_{n_j}) \leq \limsup_{j\to\infty} [(f_i(u_{n_j}) - f_i^{\delta_{n_j}}(u_{n_j})) + f_i^{\delta_{n_j}}(u_{n_j})] \leq 0,$$

i.e. the limit point $\bar{u}$ belongs to Q, which means property b.

From Theorem 2, Statement 2 and Lemma 2 it follows

Corollary 5. The iterative method $u^{k+1} = S(u^k)$, where the operator S is defined (3.3), (3.4) is robust to perturbations of the admissible set Q from (1.2) in the form (3.1),(3.8). This guarantees the convergence of the underlying iterative process (2.4) if the metric projection $P_{\tilde{Q}}$ in the process (2.4) is implemented with using the process $u^{k+1} = S(u^k)$. with error estimate (2.7).

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