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BEST APPROXIMATION OF FUNCTIONS IN THE HARDY SPACE $H_{q,\rho}$ $(1 \le q \le \infty, 0 < \rho < R)$

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Abstract. We solve extremal problems of best polynomial approximation of functions which lie the Hardy space $H_{q,R} := H_q(U_R), 1 \le q \le \infty$ and are analytic in the disk $U_R := \{z \in \mathbb{C} : |z| < R\}$.

Let $H^{(r)}_{q,R} := \{f \in H_{q,R} : \|f^{(r)}\|_{q,R} < \infty\}$ and $W^{(r)}_{q,R} := \{f \in H^{(r)}_{q,R} : \|f^{(r)}\|_{q,R} \le 1\}$ $(r \in \mathbb{Z}_+,\ 1 \le q \le \infty,\ R > 0)$. We obtain sharp inequalities between the best polynomial approximation of a function $f \in H_{q,\rho}$ $(1 \le q \le \infty,\ 0 < \rho < R)$ analytic in the disk U_R and the averaged modulus of smoothness of boundary values of the rth derivatives $f^{(r)} \in H_{q,R}$. For the class $W^{(r)}_{q,R}$, we find exact values of the supremum of the approximation. In addition, we find exact values of various n-widths in the norm of the space $H_{q,\rho}$ $(1 \le q \le \infty,\ 0 < \rho \le R)$. for the class $W^{(r)}_{q,R}(\Phi)$, which consists of all functions $f \in H^{(r)}_{q,R}$ such that, for all $k \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $k > r$, the averaged moduli of smoothness of the boundary values of the rth derivative $f^{(r)}$ which are majorized, at a given system of points $\{\pi/(2k)\}_{k\in\mathbb{N}}$, by a given majorant Φ, satisfy the condition

$$\frac{k}{\pi - 2}\int_0^{\pi/(2k)} \omega_2\big(f^{(r)}, 2t\big)_{q,R}\, dt \le \Phi\left(\frac{\pi}{2k}\right),\ k \in \mathbb{N}.$$

Key words: best polynomial approximation, Hardy space, modulus of smoothness, majorant, n-width.

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1 Introduction and main results

The problem of evaluation of exact values of n-widths of classes if functions analytic in the unit disk has been extensively studied (see, e.g., [1]– [20]). Tikhomirov [1] (the case $(q = \infty)$) and Taikov [2] (the case $(1 \le q < \infty)$) were the first to evaluate the Kolmogorov n-widths in the Hardy space H_q $(1 \le q \le \infty)$. Earlier, Babenko [3] obtained a linear method for approximation of one class of functions which are analytic in the unit disk. This method, which can be also used for delivering upper estimates of widths, was employed in [1] and [2], and also in many other studies. Later, this approach was developed by Taikov [4–6] and others (see, e.g., [7–14, 16–20]).

The purpose of the present paper is to obtain new exact values of n-widths of classes of functions analytic in the disk of radius $R \ge 1$.

Let us introduce the necessary notation and definitions. Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{R}$, and $\mathbb{C}$ be, respectively, the sets of natural numbers, nonnegative integer numbers, nonngative numbers, positive numbers, and complex numbers; $U_R := \{z \in \mathbb{C} : |z| < R\}$ be the disk of radius R in the complex plane $\mathbb{C}$, $A(U_R)$ be the set of functions analytic in the disk U_R. For an arbitrary function $f \in A(U_R)$, and $0 < \rho < R$, we set

$$M_q(f,\rho) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f\left(\rho e^{it}\right)\right|^q dt\right)^{1/q} & \text{if } 1 \le q < \infty; \\ \max\limits_{0\le t<2\pi} \left|f\left(\rho e^{it}\right)\right| & \text{if } q = \infty, \end{cases}$$

where the integral is understood in the Lebesgue sense.

Let $H_{q,R}$ $(1 \le q \le \infty,\ R \in \mathbb{R}_+)$ be the Banach Hardy space consisting of the functions $f \in A(U_R)$ with finite norm $\|f\|_{q,R} := \|f\|_{H_{q,R}} = \lim\limits_{\rho\to R-0} M_q(f,\rho)$.

It is well known (see [21, p. 279]) that the angular values $f(Re^{it}) \in L_q[0,2\pi]$ $(1 \le q \le \infty)$ exist almost everywhere on the circle $|\zeta| = R$. The norm in $H_{q,R}$ is defined by

$$\|f\|_{q,R} = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \left|f\left(Re^{it}\right)\right|^q dt\right)^{1/q}, & 1 \le q < \infty; \\ \operatorname{ess\,sup}\left\{\left|f\left(Re^{it}\right)\right| :\ 0 \le t < 2\pi\right\}, & q = \infty. \end{cases}$$

In the case $R = 1$, we set $U := U_1$, $H_q = H_{q,1}$, and define $\|f\|_q := \|f\|_{q,1}$.

For $r \in \mathbb{Z}_+$, we define $H_{q,R}^{(r)} := \{f \in A(U_R) : f^{(r)} \in H_{q,R}\}$ $(H_{q,R}^{(0)} \equiv H_{q,R})$, $f^{(r)}(z) = \frac{d^r f(z)}{dz^r}$. In what follows, we set $\alpha_{n,r} := n(n-1)\cdots(n-r+1)$, $n > r$, $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $\alpha_{n,0} = 1$. Let

$$\mathscr{P}_n := \left\{p_n(z) :\ p_n(z) = \sum_{k=0}^{n} a_k z^k,\ a_k \in \mathbb{C}\right\}$$

be the set of all complex algebraic polynomials of degree $\le n$.

The best approximation of a function $f \in H_{q,\rho}$ in the metric of the space $H_{q,\rho}$ $(1 \le q \le \infty,\ 0 < \rho < R)$. is defined by $E_{n-1}(f)_{q,\rho} := \inf\left\{\|f - p_{n-1}\|_{q,\rho} : p_{n-1} \in \mathscr{P}_{n-1}\right\}$.

Theorem 1.1. *For $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, $0 < \rho < R$, $1 \le q \le \infty$,*

$$E_{n-1}(f)_{q,\rho} \le R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}} E_{n-r-1}\left(f^{(r)}\right)_{q,R}; \tag{1}$$

this inequality becomes an equality for the function $f_0(z) = az^n$, $a \in \mathbb{C}$.

Proof. Consider $f \in H_{q,R}^{(r)}$. Let $p_{n-r-1}\left(f^{(r)}, z\right)$ be a polynomial of best $H_{q,R}$-approximation of the derivative $f^{(r)}$,

$$E_{n-r-1}\left(f^{(r)}\right)_{q,R} = \left\|f^{(r)} - p_{n-r-1}\left(f^{(r)}\right)\right\|_{q,R}. \tag{2}$$

Let $Q\left(Re^{it}\right)$ be the angular boundary values of the function

$$Q(z) := Q\left(f^{(r)}, z\right) = f^{(r)}(z) - p_{n-r-1}\left(f^{(r)}, z\right), \quad |z| \le R,$$

on the circle $|\zeta| = R$. It is easily checked that (see [22], formula (2.2), the case $s = 0$):

$$f(z) - p_{n-1}(z) = \frac{z^r}{2\pi i} \int_{|\zeta|=R} \left(\frac{z}{\zeta}\right)^{n-r} Q(\zeta) \left\{\frac{1}{\alpha_{n,r}} + 2Re \sum_{k=1}^{\infty} \frac{1}{\alpha_{n+k,\,r}} \left(\frac{z}{\zeta}\right)^k\right\} \frac{d\zeta}{\zeta}, \tag{3}$$

where $p_{n-1}(z) := p_{n-1}(f,z)$ is some polynomial from $\mathscr{P}_{n-1}$ depending on a function $f \in H^{(r)}_{q,R}$. Putting $z=\rho e^{it}$, $\zeta = Re^{i\theta}$, $0<\rho<R$, $0\le t$, $\theta \le 2\pi$ in (3), we can write this equality as

$$\begin{aligned} & f\big(\rho e^{it}\big) - p_{n-1}\big(\rho e^{it}\big) \\ = \ & R^r\left(\frac{\rho}{R}\right)^n \frac{e^{irt}}{2\pi}\int_0^{2\pi} e^{i(n-r)(t-\theta)} Q\big(Re^{i\theta}\big)\left\{\frac{1}{\alpha_{n,r}} + 2\sum_{k=1}^{\infty}\frac{(\rho/R)^k}{\alpha_{n+k,r}}\cos k(t-\theta)\right\} d\theta \\ = \ & R^r\left(\frac{\rho}{R}\right)^n \frac{e^{irt}}{2\pi}\int_0^{2\pi} e^{i(n-r)\tau} Q\big(Re^{i(t-\tau)}\big)\left\{\frac{1}{\alpha_{n,r}} + 2\sum_{k=1}^{\infty}\frac{(\rho/R)^k}{\alpha_{n+k,r}}\cos k\tau\right\} d\tau. \qquad (4)\end{aligned}$$

Applying the Abel transformation two times, we see that, for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ and $\rho \in (0,R)$, the function

$$\Phi_{n,r}(\tau) := \Phi_{n,r}(\rho/R,\tau) = \frac{1}{\alpha_{n,r}} + \sum_{k=1}^{\infty}\frac{(\rho/R)^k}{\alpha_{n+k,r}}\cos k\tau$$

is nonnegative and integrable on the interval $[0,2\pi]$ (see, e.g., [12, p. 231]), and

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi_{n,r}(\tau)d\tau = \frac{1}{\alpha_{n,r}}. \qquad (5)$$

From (4) we have

$$\begin{aligned} E_{n-1}(f)_{q,\rho} & \le \left\{\frac{1}{2\pi}\int_0^{2\pi}\big|f\big(\rho e^{it}\big) - p_{n-1}\big(\rho e^{it}\big)\big|^q dt\right\}^{1/q} \\ & = R^r\left(\frac{\rho}{R}\right)^n\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{e^{irt}}{2\pi}\int_0^{2\pi} e^{i(n-r)\tau}Q\big(Re^{i(t-\tau)}\big)\Phi_{n,r}(\tau)\,d\tau\right|^q dt\right\}^{1/q}. \qquad (6)\end{aligned}$$

Applying the generalized Minkowski inequality [23, p. 299] to the right-hand side of inequality (6) and using (5), we obtain

$$\begin{aligned} E_{n-1}(f)_{q,\rho} & \le R^r\left(\frac{\rho}{R}\right)^n\left\{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\big|Q\big(Re^{i(t-\tau)}\big)\big|\big|\Phi_{n,r}(\tau)\big|d\tau\right)^q dt\right\}^{1/q} \\ & \le R^r\left(\frac{\rho}{R}\right)^n\left(\frac{1}{2\pi}\int_0^{2\pi}\big|Q\big(Re^{it}\big)\big|^q dt\right)^{1/q}\left(\frac{1}{2\pi}\int_0^{2\pi}\big|\Phi_{n,r}(\tau)\big|d\tau\right) \\ & = R^r\left(\frac{\rho}{R}\right)^n\|Q\|_{q,R}\left(\frac{1}{2\pi}\int_0^{2\pi}\big|\Phi_{n,r}(\tau)\big|d\tau\right) = R^r\left(\frac{\rho}{R}\right)^n\|Q\|_{q,R}\frac{1}{\alpha_{n,r}}. \qquad (7)\end{aligned}$$

Next, $\|Q\|_{q,R} = E_{n-r-1}\big(f^{(r)}\big)_{q,R}$ by (2), and now from (7) we finally have

$$E_{n-1}(f)_{q,\rho} \le R^r\left(\frac{\rho}{R}\right)^n\big\|Q\big\|_{q,R}\cdot\frac{1}{\alpha_{n,r}} \le R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}E_{n-r-1}\big(f^{(r)}\big)_{q,R},$$

which proves inequality (1).

Let us show that inequality (1) becomes an equality for the function

$$f_0(z) = az^n \in H^{(r)}_{q,\rho} \quad (1\le q\le\infty,\ 0<\rho<R),\ a\in\mathbb{C},\ n\in\mathbb{N},\ r\in\mathbb{Z}_+,\ n>r, \qquad (8)$$

For this function and its rth derivative, we have

$$E_{n-1}(f_0)_{q,\rho} = |a|\rho^n, \quad E_{n-r-1}\big(f_0^{(r)}\big)_{q,R} = |a|\alpha_{n,r}R^{n-r}. \qquad (9)$$

Hence

$$R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}E_{n-r-1}\big(f_0^{(r)}\big)_{q,R} = |a|\rho^n = E_{n-1}(f_0)_{q,\rho}.$$

This proves Theorem 1.1. □

Let $W^{(r)}H_{q,R} := \left\{ f \in H_{q,R}^{(r)} : \ \|f^{(r)}\|_{q,R} \le 1 \right\}$ $(1 \le q < \infty,\ R \in \mathbb{R}_+)$.

Theorem 1.2. *For all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ and for all $1 \le q \le \infty$, $0 < \rho < R$,*

$$E_{n-1}\big(W^{(r)}H_{q,R}\big)_{q,\rho} := \sup\left\{E_{n-1}(f)_{q,\rho} : f \in W^{(r)}H_{q,R}\right\} = R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}}. \tag{10}$$

Proof. For each function $f \in W^{(r)}H_{q,R}$, the best approximation of its derivative $f^{(r)}$ satisfies $E_{n-r-1}\big(f^{(r)}\big)_{q,R} \le \big\|f^{(r)}\big\|_{q,R} \le 1$, and hence from inequality (1) we have the following upper bound for the quantity on the left of (10):

$$E_{n-1}\big(W^{(r)}H_{q,R}\big)_{q,\rho} \le R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}}. \tag{11}$$

To obtain a similar lower estimate, we will use the function $f_1(z) = z^n/(R^{n-r}\alpha_{n,r})$ from the class $W^{(r)}H_{q,R}$. We have $E_{n-1}(f_1)_{q,\rho} = R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}}$, and hence the following lower estimate holds:

$$E_{n-1}\big(W^{(r)}H_{q,R}\big)_{q,\rho} \ge E_{n-1}(f_1)_{q,\rho} = R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}}. \tag{12}$$

Combining inequalities (11) and (12), we get the required equality (10). This completes the proof of Theorem 1.2. □

2 Estimate of the best approximation $E_{n-1}(f)_{q,\rho}$ in terms of the averaged modulus of smoothness $\omega_2\big(f^{(r)}, t\big)_{q,R}$

For an arbitrary function $f \in H_{q,R}^{(r)}$, the modulus of smoothness of the derivative $f^{(r)}$ is defined by

$$\omega_2\big(f^{(r)}, 2t\big)_{q,R} := \sup_{|h|\le t} \big\| f^{(r)}\big(Re^{i(\cdot+h)}\big) - 2f^{(r)}\big(Re^{i(\cdot)} + f\big(Re^{i(\cdot-h)}\big)\big\|_{q,R}.$$

We have the following result:

Theorem 2.1. *Let $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, $1 \le q \le \infty$, $0 < \rho < R$. Then, for an arbitrary function $f \in H_{q,R}^{(r)}$,*

$$E_{n-1}(f)_{q,\rho} \le R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}} \frac{n-r}{\pi-2} \int_0^{\pi/2(n-r)} \omega_2\big(f^{(r)}, 2t\big)_{q,R} dt; \tag{13}$$

this inequality becomes an equality for the function $f_0(z) = az^n$, $a \in \mathbb{C}$, $n \in \mathbb{N}$.

Proof. From Theorem 1 of [4] it follows that, for an arbitrary function $f \in H_{q,R}^{(r)}$,

$$E_{n-1}(f)_{q,R} \le \frac{n}{\pi-2} \int_0^{\pi/(2n)} \omega_2\big(f, 2t\big)_{q,R} dt,\ n \in \mathbb{N}; \tag{14}$$

moreover, this inequality becomes an equality for the function $f_0(z) = az^n$, $a \in \mathbb{C}$, $n \in \mathbb{N}$. If in inequality(14) we replace n by $n-r$ and replace the function f by the derivative $f^{(r)}$, we have

$$E_{n-r-1}\big(f^{(r)}\big)_{q,R} \le \frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)} \omega_2\big(f^{(r)},2t\big)_{q,R}dt,\ n\in\mathbb{N},\ r\in\mathbb{Z}_+,\ n>r. \tag{15}$$

Using inequality (15), and employing (1), we have

$$E_{n-1}(f)_{q,\rho} \le R^r\left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}} E_{n-r-1}\big(f^{(r)}\big)_{q,R} \le R^r\left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}} \frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)} \omega_2\big(f^{(r)},2t\big)_{q,R}dt,$$

which proves inequality (13).

Let us prove that (13) becomes an equality for the function $f_0(z) = az^n$, $a\in\mathbb{C}$, $n\in\mathbb{N}$. We have

$$f_0^{(r)}(z) = a\alpha_{n,r}z^{n-r}, \quad \omega_2\big(f_0^{(r)},2t\big)_{q,R} = |a|\alpha_{n,r}R^{n-r}2(1-\cos 2(n-r)t),$$

$$E_{n-1}(f_0)_{q,\rho} = |a|\rho^n, \quad \int_0^{\pi/2(n-r)} \omega_2\big(f_0^{(r)},2t\big)_{q,R}dt = |a|\alpha_{n,r}R^{n-r}\frac{\pi-2}{n-r},$$

and hence,

$$R^r\left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}} \frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)} \omega_2\big(f_0^{(r)},2t\big)_{q,R}dt = |a|\rho^n = E_{n-1}(f_0)_{q,\rho},$$

which completes the proof of Theorem 2.1 □

3 Exact values of the n-width of the class $W_{q,R}^{(r)}(\Phi)$ $(r\in\mathbb{Z}_+,\ 1\le q<\infty,\ R\ge 1)$ in the space $H_{q,\rho}$ $(1\le q<\infty,\ 0<\rho\le R,\ R\ge 1)$

Let S be the unit ball in $H_{q,\rho}$, let $\mathfrak{M}$ be some convex centrally symmetric subset of $H_{q,\rho}$, let $\mathscr{L}_n\subset H_{q,\rho}$ be an n-dimensional linear subspace, $\mathscr{L}^n\subset H_{q,\rho}$ be a subspace of codimension n, and let $A: H_{q\rho}\to\mathscr{L}_n$ be a linear continuous operator which maps $H_{q,\rho}$ into $\mathscr{L}_n$.

The one-sided approximation of a set $\mathfrak{M}\subset H_{q,\rho}$ by a subspace $\mathscr{L}_n$ of the space $H_{q,\rho}$ is defined by $E_n(\mathfrak{M})_{q,\rho} := E(\mathfrak{M},\mathscr{L}_n)_{q,\rho} = \sup\big\{\inf\{\|f-\varphi\|_{q,\rho}:\varphi\in\mathscr{L}_n\}:f\in\mathfrak{M}\big\}$. The quantity

$$\mathscr{E}_n(\mathfrak{M})_{q,\rho} := \mathscr{E}(\mathfrak{M},\mathscr{L}_n)_{q,\rho} = \inf\big\{\sup\{\|f-A(f)\|_{q,\rho}:f\in\mathfrak{M}\}:AH_{q,\rho}\subset\mathscr{L}_n\big\} \tag{16}$$

characterizes the best linear approximation of a set $\mathfrak{M}$ by elements of the subspace $\mathscr{L}_n\subset H_{q,\rho}$. A linear operator A^*, $A^*H_{q,\rho}\subset\mathscr{L}_n$ (if exists) for which the infimum in (16) is attained, i.e., $\mathscr{E}(\mathfrak{M},\mathscr{L}_n)_{q,\rho} = \sup\Big\{\big\|f-A^*(f)\big\|_{q,\rho}:f\in\mathfrak{M}\Big\}$, is a best linear approximation method for $\mathfrak{M}$. The quantities

$$\begin{aligned}
b_n(\mathfrak{M},H_{q,\rho}) &= \sup\big\{\sup\{\varepsilon>0:\varepsilon S\cap\mathscr{L}_{n+1}\subset\mathfrak{M}\}:\mathscr{L}_{n+1}\subset H_{q,\rho}\big\},\\
d_n(\mathfrak{M},H_{q,\rho}) &= \inf\big\{E(\mathfrak{M},\mathscr{L}_n)_{q,\rho}:\mathscr{L}_n\subset H_{q,\rho}\big\},\\
d^n(\mathfrak{M},H_{q,\rho}) &= \inf\big\{\sup\{\|f\|_{q,\rho}:f\in\mathfrak{M}\cap\mathscr{L}^n\}:\mathscr{L}^n\subset H_{q,\rho}\big\},\\
\delta_n(\mathfrak{M},H_{q,\rho}) &= \inf\big\{\mathscr{E}(\mathfrak{M},\mathscr{L}_n)_{q,\rho}:\mathscr{L}_n\subset H_{q,\rho}\big\}
\end{aligned}$$

are, respectively, the *Bernstein, Kolmogorov, Gelfand, and linear n-widths* (see, e.g. [12, Ch. II, §1–4], [24, Ch. III, §1], and [23, Ch. I, §1.3]).

Let us recall that the above n-widths satisfy the following relations (see [12, 24]):

$$b_n(\mathfrak{M}, H_{q,\rho}) \le \begin{matrix} d^n(\mathfrak{M}, H_{q,\rho}) \\ d_n(\mathfrak{M}, H_{q,\rho}) \end{matrix} \le \delta_n(\mathfrak{M}, H_{q,\rho}) \tag{17}$$

In what follows, we assume that $\Phi(t)$, $t \ge 0$ is an arbitrary continuous increasing function such that $\Phi(0) = 0$. Using Φ as a majorant, consider the following classes of Taikov functions [4]:

$$W_{q,R}^{(r)}(\Phi) := \left\{ f \in H_{q,R}^{(r)} : \frac{k}{\pi - 2} \int_0^{\pi/(2k)} \omega_2\big(f^{(r)}, 2t\big) dt \le \Phi\left(\frac{\pi}{2k}\right),\ k \in \mathbb{N} \right\},$$

where $r \in \mathbb{Z}_+$, $1 \le q \le \infty$, $R \ge 1$. In the case $R = 1$, we set $W_q^{(r)}(\Phi) := W_{q,1}^{(r)}(\Phi)$. In [4], it was shown that if the majorant $\Phi(t)$ for $0 < t \le \pi/2$ satisfies the constraint

$$\frac{\Phi(\lambda t)}{\Phi(t)} \ge \frac{\pi}{\pi - 2} \begin{cases} 1 - \dfrac{2}{\lambda\pi} \sin \dfrac{\lambda\pi}{2} & \text{if } 0 < \lambda \le 2, \\ 2\left(1 - \dfrac{1}{\lambda}\right) & \text{if } \lambda \ge 2, \end{cases} \tag{18}$$

then, for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ for $1 \le q \le \infty$,

$$d_n\big(W_q^{(r)}(\Phi), H_q\big) = \frac{1}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)}\right). \tag{19}$$

Taikov [4] also proved that constraint (18) is satisfied, e.g., by the function $\Phi_*(t) = t^{2/(\pi-2)}$.

Following [4, 20], it would be interesting to find sharp values of the above n-widths of the classes $W_{q,R}^{(r)}(\Phi)$, $r \in \mathbb{Z}_+$, $1 \le q < \infty$, $R \ge 1$ in the space $H_{q,\rho}$, $1 \le q < \infty$, $0 < \rho \le R$.

We have the following result:

Theorem 3.1. *Let $r \in \mathbb{Z}_+$, $1 \le q \le \infty$, $R \ge 1$ and let a majorant Φ satisfy constraint* (18). *Then, for each $n \in \mathbb{N}$, $n > r$,*

$$\lambda_n\big(W_{q,R}^{(r)}(\Phi), H_{q,\rho}\big) = E_{n-1}\big(W_{q,R}^{(r)}(\Phi)\big)_{q,\rho} = \mathscr{E}_{n-1}\big(W_{q,R}^{(r)}(\Phi)\big)_{q,R} = R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)}\right), \tag{20}$$

where $\lambda_n(\cdot)$ is any of the n-widths $b_n(\cdot)$, $d_n(\cdot)$, $d^n(\cdot)$, $\delta_n(\cdot)$.

Proof. By the definition of the class $W_{q,R}^{(r)}(\Phi)$, from inequality (13) and (17), we have

$$\begin{aligned} & b_n\big(W_{q,R}^{(r)}(\Phi), H_{q,\rho}\big) \le d_n\big(W_{a,R}^{(r)}(\Phi), H_{q,\rho}\big) \le E_n\big(W_{q,R}^{(r)}(\Phi)\big)_{q,\rho} \\ = \ & \sup\big\{E_{n-1}(f)_{q,\rho} : f \in W_{q,R}^{(r)}(\Phi)\big\} \le R^r \left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}} \Phi\left(\frac{\pi}{2(n-r)}\right), \end{aligned} \tag{21}$$

which gives the required upper estimate for the *Bernstein* and *Kolmogorov* n-width. To obtain a lower estimate, we use the subspace of complex algebraic polynomials of degree $\le n$. It is known that, for an arbitrary polynomial $p_n \in \mathscr{P}_n$, we have the inequality (see [12, Ch. III, §2])

$$\big\|p_n^{(r)}\big\|_{q,R} \le R^{n-r} \alpha_{n,r} \|p_n\|_q, \tag{22}$$

where $r \in \mathbb{Z}_+$, $n \in \mathbb{N}$, $n > r$, $1 \le q \le \infty$, $0 < \rho \le R$, $R \ge 1$. Using the inequality $\|p_n\|_q \le \rho^{-n}\|p_n\|_{q,\rho}$, which was proved by E. Hill., G. Szegő, and Ya. D. Tamarkin (see, e.g., [26]) from (22), we find that

$$\|p_n^{(r)}\|_{q,R} \le \frac{1}{R^r}\left(\frac{R}{\rho}\right)^n \alpha_{n,r}\|p_n\|_{q,\rho}. \tag{23}$$

Consider the ball $S_{n+1} := \left\{p_n \in \mathscr{P}_n : \|p_n\|_{q,\rho} \le R^r\left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right)\right\}$. We set $\Delta_n(t) := \{2(1-\cos nt)\ \text{ if }\ 0 \le t \le \pi/n;\ \ 4\ \text{ if }\ t \ge \pi/n\}$. From the inequality

$$\|p_n(ze^{it}) - 2p_n(z) + p_n(ze^{-it})\|_{q,R} \le \Delta_n(t)\|p_n\|_{q,R},$$

which follows from one result of [4], for an arbitrary polynomial $p_n \in \mathscr{P}_n$, we have

$$\omega_2(p_n, 2t)_{q,R} \le \Delta_n(t)\|p_n\|_{q,R}, \quad t \ge 0. \tag{24}$$

We claim that the ball S_{n+1} lies in the class $W_{q,R}^{(r)}(\Phi)$. There are two cases to consider: $2k \ge n-r$ and $2k \le n-r$.

Let first $2k \ge n-r$. Then, for an arbitrary polynomial $p_n \in S_{n+1}$, from (24) and (23) we have

$$\begin{aligned}
&\frac{k}{\pi-2}\int_0^{\pi/(2k)} \omega_2\big(p_n^{(r)}, 2t\big)_{q,R}dt \le \|p_n^{(r)}\|_{q,R}\frac{2k}{\pi-2}\int_0^{\pi/(2k)}\big(1-\cos(n-r)t\big)dt \\
\le\ & \frac{1}{R^r}\left(\frac{R}{\rho}\right)^n \alpha_{n,r}\|p_n\|_{q,\rho}\frac{\pi}{\pi-2}\left(1 - \frac{2k}{\pi(n-r)}\sin\frac{\pi(n-r)}{2k}\right) \\
\le\ & \frac{\pi}{\pi-2}\left(1 - \frac{2k}{\pi(n-r)}\sin\frac{\pi(n-r)}{2k}\right)\Phi\left(\frac{\pi}{2(n-r)}\right).
\end{aligned} \tag{25}$$

Using the first inequality from constraint (18) and putting

$$x = \frac{\pi}{2(n-r)}, \qquad \lambda = \frac{n-r}{k}, \qquad \lambda x = \frac{\pi}{2k} \tag{26}$$

in (25), we get

$$\frac{k}{\pi-2}\int_0^{\pi/(2k)} \omega_2\big(p_n^{(r)}, 2t\big)_{q,R}dt \le \Phi\left(\frac{\pi}{2k}\right). \tag{27}$$

Now let $2k \le n-r$. In this case, invoking again inequalities (24) and (23), we have, for each $p_n \in S_{n+1}$,

$$\begin{aligned}
&\frac{k}{\pi-2}\int_0^{\pi/(2k)} \omega_2\big(p_n^{(r)}, 2t\big)_{q,R}dt \\
\le\ & \Phi\left(\frac{\pi}{2(n-r)}\right)\frac{2k}{\pi-2}\left(\int_0^{\pi/(2k)}\big(1-\cos(n-r)\big)tdt + \int_{\pi/(n-r)}^{\pi/(2k)} 2dt\right) \\
=\ & \frac{2\pi}{\pi-2}\left(1 - \frac{k}{n-r}\right)\Phi\left(\frac{\pi}{2(n-r)}\right).
\end{aligned} \tag{28}$$

Using notation (26) and the second inequality from (18), we get (27) from (28). This proves the inclusion $S_{n+1} \in W_{q,R}^{(r)}(\Phi)$. Now, by the definition of the *Bernstein* n-width, we have

$$b_n\big(W^{(r)}_{q,R}(\Phi), H_{q,\rho}\big) \geq b_n\big(S_{n+1}, H_{q,\rho}\big) = R^r\left(\frac{\rho}{R}\right)^n \frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right). \qquad (29)$$

Combining inequalities (21) and (29), we find that

$$b_n\big(W^{(r)}_{q,R}(\Phi),H_{q,\rho}\big)=d_n\big(W^{(r)}_{q,R}(\Phi),H_{q,\rho}\big)=E_n\big(W^{(r)}_{q,R}(\Phi)\big)_{q,\rho}=R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right). \quad (30)$$

□

The next lemma is required for a similar upper bound for the linear n-width.

Lemma 3.1. *For an arbitrary function* $f(z)=\sum\limits_{k=0}^{\infty}c_k(f)z^k$ *from the class* $W^{(r)}_{q,R}(\Phi)$, *for all* $r\in\mathbb{Z}_+$, $1\leq q\leq\infty$, $R\geq 1$, $n\in\mathbb{N}$, $n>r$,

$$\|f-\Lambda_{n-1,r}(f)\|_{q,\rho} \leq R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right), \qquad (31)$$

where the linear polynomial operator $\Lambda_{n-1,r}(f)$ *has the form*

$$\Lambda_{n-1,r}(f,z)=\sum_{k=0}^{r-1}c_k(f)z^k+\sum_{k=r}^{n-1}\left\{1+\frac{\alpha_{n,r}}{\alpha_{2n-1,r}}\left[\sigma_{k,r}\left(1-\left(\frac{k-r}{2n-r-k}\right)^2\right)-1\right]\left(\frac{|z|}{R}\right)^{2(n-k)}\right\}c_kz^k, \quad (32)$$

$$\sigma_{k,r} := \frac{2(n-r)}{\pi-2}\int_0^{\pi/2(n-r)}\big(1-\sin(n-r)x\big)\cos(k-r)x\,dx,\ \ k\geq r,\ k\in\mathbb{N}.$$

If the majorant $\Phi(t)$ *for* $0<t\leq\pi/2$ *satisfies constraint* (18), *then estimate* (31) *is sharp in the sense that there exists a function* $g\in W^{(r)}_{q,R}(\Phi)$, *for which inequality* (31) *becomes an equality.*

Proof. In [20, formula (22)], it was shown that, for any function $f\in H^{(r)}_{q,R}$ ($r\in\mathbb{Z}_+$, $1\leq q\leq\infty$, $0<\rho<R$, $R\geq 1$),

$$f\big(\rho e^{it}\big)-\Lambda_{n-1,r}\big(f,\rho e^{it}\big)=\frac{1}{2\pi}\int_0^{2\pi}\Big[f^{(r)}\big(R^{i\theta}\big)-V_{n-r-1,2}\big(\mathscr{F}^{(r)},R^{i\theta}\big)\Big]e^{ir\theta}G_R(\rho,t-\theta)d\theta, \quad (33)$$

where

$$\mathscr{F}\big(f^{(r)};z\big)=\frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)}\Big[f^{(r)}\big(ze^{ix}\big)+f^{(r)}\big(ze^{-ix}\big)\Big](1-\sin(n-r))x\,dx=\sum_{k=r}^{\infty}\delta_{k,r}\alpha_{k,r}c_k(f)z^{k-r}, \quad (34)$$

$$V_{n-r-1,2}\big(\mathscr{F}^{(r)},z\big)=\sum_{k=0}^{n-r-1}\delta_{k+r,r}\alpha_{k+r,r}c_{k+r}(f)\left(1-\left(\frac{k}{2(n-r)-k}\right)^2\right)z^k,$$

$$G_R(\rho,t)=R^r\left(\frac{\rho}{R}\right)^n e^{irt}\Phi_{n,r}(t). \qquad (35)$$

By the definition of the norm in the space $H_{q,\rho}$, we have from (33)

$$\big\|f-\Lambda_{n-1,r}(f)\big\|_{q,\rho}$$

$$=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi}\int_0^{2\pi}\Big[f^{(r)}\big(Re^{i\theta}\big)-V_{n-r-1,2}\big(\mathscr{F}\big(f^{(r)}\big),Re^{i\theta}\big)\Big]e^{ir\theta}G_R(\rho,t-\theta)d\theta\right|^q dt\right)^{1/q}. \quad (36)$$

Applying the generalized Minkowski inequality to the right-hand side of (36) and using the equality

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|G_R(\rho,t)\Big|dt = R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}},$$

which follows from (35) and (5), we get

$$\begin{aligned}
&\big\|f-\Lambda_{n-1,r}(f)\big\|_{q,\rho}\\
\le\;&\left(\frac{1}{2\pi}\int_0^{2\pi}\Big|f^{(r)}\big(Re^{i\theta}\big)-V_{n-r-1,2}\big(\mathscr{F}\big(f^{(r)}\big),Re^{i\theta}\big)\Big|^q dt\right)^{1/q}\left(\frac{1}{2\pi}\int_0^{2\pi}|G_R(\rho,t)|dt\right)\\
=\;&R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\big\|f^{(r)}-V_{n-r-1,2}\big(\mathscr{F}(f^{(r)})\big)\big\|_{q,R}. \qquad (37)
\end{aligned}$$

Next, we have

$$\big\|f^{(r)}-V_{n-r-1,2}\big(\mathscr{F}(f^{(r)})\big)\big\|_{q,R}\le\big\|f^{(r)}-\mathscr{F}(f^{(r)})\big\|_{q,R}+\big\|\mathscr{F}\big(f^{(r)}\big)-V_{n-r-1,2}\big(\mathscr{F}(f^{(r)})\big)\big\|_{q,R}, \quad (38)$$

and hence, by inequalities (28) and (29) in [20, p.12],

$$\big\|f^{(r)}-\mathscr{F}(f^{(r)})\big\|_{q,R}\le\frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)}\omega_2\big(f^{(r)},2x\big)_{q,R}\big(1-\sin(n-r)x\big)dx, \quad (39)$$

$$\big\|\mathscr{F}\big(f^{(r)}\big)-V_{n-r-1}\big(\mathscr{F}(f^{(r)})\big)\big\|_{q,R}\le\frac{n-r}{\pi-2}\int_0^{\pi/2(n-r)}\omega_2\big(f^{(r)},2x\big)_{q,R}\sin(n-r)x\,dx, \quad (40)$$

Now by (37)–(40), for an arbitrary function $f\in W^{(r)}_{q,R}(\Phi)$, we have

$$\big\|f-\Lambda_{n-1,r}(f)\big\|_{q,\rho}\le R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\frac{n-r}{\pi-2}\int\limits_0^{\pi/2(n-r)}\omega_2\big(f^{(r)},2x\big)_{q,R}dx\le R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right)$$

which proves inequality (31).

We claim that if the majorant Φ satisfies constraint (18), then the class $W^{(r)}_{q,R}(\Phi)$, contains a function for which inequality (31) becomes an equality. To this end, consider the function $f_2(z)=\frac{1}{R^{n-r}\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right)z^n$. We have $\|f_2\|_{q,\rho}=R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right)$, and hence the function f_2 lies in the ball S_{n+1}.

Therefore, $f_2\in W^{(r)}_{q,R}(\Phi)$, and further, since $\Lambda_{n-1,r}(f_2)\equiv 0$ by (32), we have

$$\big\|f_2-\Lambda_{n-1,r}(f_2)\big\|_{q,\rho}=\big\|f_2\big\|_{q,\rho}=R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right). \quad (41)$$

Equality (41) means that the linear polynomial operator (32) is a best linear approximation method of the class $W^{(r)}_{q,R}(\Phi)$ in the space $H_{q,\rho}$ $(1\le q<\infty,\ 0<\rho<R,\ R\ge 1)$. From (31) and (41) it follows that if the majorant Φ satisfies (18), then

$$\delta_n\big(W^{(r)}_{q,R}(\Phi),H_{q,\rho}\big)\le\mathscr{E}\big(W^{(r)}_{q,R}(\Phi):\Lambda_{n-1,r}\big)=R^r\left(\frac{\rho}{R}\right)^n\frac{1}{\alpha_{n,r}}\Phi\left(\frac{\pi}{2(n-r)}\right). \quad (42)$$

By inequalities (17) for n-widths, from (30) and (42) we get the required equality (20). This completes the proof of Theorem 3.1. □

References

[1] Tikhomirov V.M., *Diameters of sets in function spaces and the theory of best approximations*, Russian Math. Surveys, 15(3), 1960, 75–111.

[2] Taikov L.V., *Best approximation in the mean of certain classes of analytic functions*, Math. Notes, 1(2), 1967, 104–109

[3] Babenko K.I., *Best approximations for a class of analytic functions*, Izv. Akad. Nauk SSSR Ser. Mat. 22(5), 1958, 631–640. (Russian).

[4] Taikov L.V., *Diameters of certain classes of analytic functions*, Math. Notes, 22(2), 1977, 650–656.

[5] Ainulloev N., Taikov L.V., *Best approximation in the sense of Kolmogorov of classes of functions analytic in the unit disc*, Math. Notes, 40(3), 1986, 699–705.

[6] Taikov L.V., *Some exact inequalities in the theory of approximation of functions*, Analysis Mathematica. 2, 1976, 77–85.

[7] Dveirin M.Z., *Widths and ε-entropy of classes of functions that are analytic in the unit circle of functions*, Function theory, functional analysis and their applications, 23, 1975, 32–46.

[8] Dveirin M.Z., Chebanenko I.V., *On polynomial approximation in the weighted Banach spaces of analytic functions*, Mapping theory and approximation of functions. Kiev: Nukova dumka, 1983, 62–73.

[9] Yu.A. Farkov, *Widths of Hardy classes and Bergman classes on the ball in $\mathbb{C}^n$*, Russian Math. Surveys, 45(5), 1990, 229–231.

[10] Farkov Yu.A., *n-Widths, Faber expansion, and computation of analytic functions*, Journal of complexity. 12(1), 1996, 58–79.

[11] Farkov Yu.A., Stessin M.I., *The n-width of the unit ball of H^q*, Journal of Approx. Theory. 67(3), 1991, 347–356.

[12] Pinkus A., *n-Widths in Approximation Theory*, Berlin. Springer-Verlag. Heidelberg. New York. Tokyo, 1985.

[13] Vakarchuk S.B. *On the widths of certain classes of functions analytic in the unit disc. I* Ukr. Matem. Journal, 42(7), 1990, 873–881.

[14] Vakarchuk S.B. *On the widths of certain classes of functions analytic in the unit disc. II*, Ukr. Matem. Journal, 42(8), 1990, 1019–1026.

[15] Vasil'eva A.A., *Kolmogorov widths of an intersection of a finite family of Sobolev classes*, Izv. Math., 88(1), 2024, 18–42.

[16] Vakarchuk S.B., *Exact values of widths of classes of analytic functions on the disk and best linear approximation methods*, Math. Notes, 72(5), 2002, 615–619

[17] Vakarchuk S.B., *On some extremal problems in the theory of approximations in the complex plane*, Ukr. Matem. Journal, 56(9), 2004, 1155–1171

[18] Shabozov M.Sh., Shabozov O.Sh., *Widths of some classes of analytic functions in the Hardy space H_2*, Math. Notes, 68(5), 2000, 675–579.

[19] Shabozov M.Sh., Yusupov G.A. *Best approximation and values of widths of some classes of analytic functions*, Dokl. RAN, 383(2), 2002, 171–174.

[20] Vakarchuk S.B., Shabozov M.Sh., *The widths of classes of analytic functions in a disc*, Sb. Math., 201(8), 2010, 1091–1110.

[21] Smirnov V.I., Lebedev N.A., *Functions of a complex variable.* Constructive theory, MIT Press, Cambridge, MA, 1968.

[22] Shabozov M.Sh., *On the best simultaneous approximation in the Bergman space* B_2, Math. Notes, 114(3), 2023, 377–386.

[23] Korneichuk N.P., *Extremal Problems of the Approximation Theory*, Izdat. Nauka, Moscow, 1976 (in Russian).

[24] Tikhomirov V.M., *Some problems of theory of approximation* (Moscow: MSU, 1976), 304. (in Russian)

[25] Taikov L.V., *Inequalities containing best approximations and the modulus of continuity of functions in* L_2, Math. Notes, 20(3), 1976, 797–800

[26] Shikhalev N.I., *An inequality of Bernstein–Markov kind for analytic functions*, Dokl. Akad. Nauk Azerb. SSR 31(8), 1975, 9–14. (Russian)

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