EURASIAN JOURNAL OF MATHEMATICAL AND COMPUTER APPLICATIONS
ISSN 2306–6172
Volume 12, Issue 3 (2024) 105 – 116

CONTINUATION PROBLEM OF THE ELECTROMAGNETIC FIELD IN THE FREQUENCY DOMAIN FOR HORIZONTALLY LAYRED MEDIA

Orman I. M., Kurmashev I. G., Karchevsky A. L.

Abstract The paper presents analytical expressions for solving the problem of continuation of the electromagnetic field in the frequency domain for a horizontally layered medium. The numerical solution algorithm uses layerwise recalculation of the required quantities, for which analytical expressions are presented in a form that allows calculations and layerwise recalculation without the accumulation of rounding errors. The solution to the continuation problem is obtained as a solution to the cost functional minimization problem. The strong convexity of the functional is proven, which implies the existence of a unique solution to the problem posed.

Key words: continuation of the electromagnetic field, inverse problem, frequency domain, horizontally layered medium.

AMS Mathematics Subject Classification: 35A24, 35A25, 35B60, 35Q61, 35R30.

DOI: 10.32523/2306-6172-2024-12-3-105-116

1 Introduction

The paper presents analytical expressions for solving the problem of continuation of the electromagnetic field in the frequency domain for a horizontally layered medium. The numerical algorithm is based on layer-by-layer recalculation. Analytical expressions are presented in a form that allows calculations and layer-by-layer recalculation without the accumulation of rounding errors.

The continuation problem to a certain depth is relevant when the electromagnetic properties of the first few layers are known, and below the electromagnetic properties of the layered medium are subject to determination. If the continuation problem is solved, then the inverse problem of determining the unknown medium parameters can be posed on a smaller domain. This will speed up the solution of the direct problem, and, consequently, the inverse problem.

As mentioned above, we will use the layer-by-layer recalculation method to obtain analytical expressions for solving the continuation problem. One of the first technologically advanced layer-by-layer recalculation algorithms for solving a boundary value problem for second-order differential equation for a horizontally layered medium was Tikhonov-Shakhsuvarov algorithm [1]. However, it had some limitations: analytical expressions contained exponential functions whose exponents had positive real parts, which led to the accumulation of rounding errors during calculations and layer-by-layer recalculation. This drawback of the proposed layer-by-layer recalculation method was eliminated in the Dmitriev's works (see, for example, [2, 3]). To construct expressions for calculations, he used the idea of Gelfand and Lokutsievsky [4] for solving a boundary value problem for second-order differential equation. It made possible to obtain expressions for calculations that are resistant to the accumulation of rounding errors. Further, this method of layer-by-layer recalculation was developed for solving systems of second-order differential equations: for a system of equations of the theory of elasticity (horizontally layered isotropic medium [5]-[7], isotropic medium with absorption [8, 9], transversely isotopic medium with a symmetry axis coinciding with the Oz axis [10], a medium of any type of anisotropy [11, 12]), for calculating the gradient of the residual functional for solving inverse problems to determine the velocity parameters of thinly-stratified layer [13, 14], for Maxwell's equations (horizontally layered medium of any type of anisotropy [15]), for the system of equations of convective heat and moisture transfer [16], for the fourth-order differential equation of transverse vibrations of a piecewise homogeneous beam [17]. Currently, the layer-by-layer recalculation method for solving a boundary value problem for second-order differential equations or systems of second-order differential equations for horizontally layered media are recognized as the most suitable for calculations [18]. In this work, the layer-by-layer recalculation method is used to find expressions that are resistant to the accumulation of rounding errors for the numerical solution of the continuation problem, which is the Cauchy problem for a second-order differential equation.

The layer-by-layer recalculation method makes it possible to obtain a numerical method for solving the direct problem that is not only resistant to rounding errors but also requires relatively little time for calculations. This is a very useful property, especially when solving inverse problems using the optimization method, since in this case solving the inverse problem is reduced to repeatedly solving the direct problem (see, for example, [19]-[33]). The time for solving the direct and, therefore, the inverse problem can be greatly reduced if it is necessary to solve the inverse problem of determining the parameters of the medium, starting from a certain depth, since the upper part of the medium is known. This could be, for example, a road surface, an airfield runway, the upper part of an archaeological section, etc. Having solved the field continuation problem to a certain depth and excluding information about the appropriate layers from the known data, we obtain a simpler formulation of the inverse problem.

The numerical solving the continuation problems is, as a rule, unstable, and therefore requires the use of regularization methods (see, for example, [38]-[44]). 2D and 3D field continuation problems in the stationary case is mathematically formulated as the Cauchy problem for an elliptic equation, in the non-stationary case – as a problem for a parabolic or hyperbolic equation with data on a timelike boundary. For numerical solving the Cauchy problem for an elliptic equation, many methods have been proposed (see, for example, the review in the work [45]). For the numerical solution of equations with data on the timelike boundary, we know only two methods – the optimization method [46] and the adjoint operator method [47]. As a rule, the proposed methods for solving the Cauchy problem (see [45]) were tested on simulated data, in the works [47]-[50] the problem was solved on data obtained during laboratory experiments. Also, the solution to a parabolic equation with data on a timelike boundary was obtained after laboratory measurements [51, 52]. Some simple formulations of field continuation problems can be found in [53]-[55].

In our case, when the continuation problem is solved in the frequency domain, it is necessary to obtain analytical expressions for solving the Cauchy problem for a second-order differential equation. This is a well-posed problem. This means that a correct method for solving this problem must be presented. Here we propose a solution method based on layer-by-layer recalculation: in each layer, analytical expressions are obtained for solving the problem such that numerical calculations and recurrent recalculation do not accumulate rounding errors.

2 Mathematical formulation of the problem

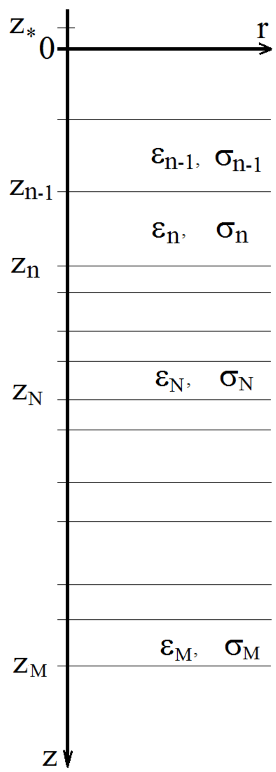

Figure 1: Medium model.

Let the medium be a horizontally layered structure with interfaces z_n $(n = \overline{0, M})$, the n-th layer is the interval $[z_{n-1}, z_n]$, thickness of the n-th layer $h_n = z_n - z_{n-1}$, $(-\infty, z_0]$ is air, $[z_M, \infty)$ is underlying the half-space (see Fig/ 1).

The propagation of the electromagnetic field is described by the Maxwell equations. The medium is characterized by permittivity ε, conductivity σ and magnetic permeability μ. Permittivity $\varepsilon = \epsilon_0 \epsilon$ ($\epsilon_0 = 8.854 \cdot 10^{-12}$(F/m) is the permittivity of vacuum and $\epsilon \geq 1$ is the relative permittivity of the medium). For most geophysical media $\mu = 4\pi \cdot 10^{-7}$(H/m). Relative permittivity of air $\epsilon = 1$, conductivity of air $\sigma = 0$. We will assume that the relative permittivity ϵ and the conductivity σ of the medium depend only on depth, i.e. $\epsilon = \epsilon(z)$ and $\sigma = \sigma(z)$, and are piecewise constant functions. The electromagnetic properties of the n-th layer are determined by the constants ϵ_n and σ_n.

Assume that the electromagnetic field is excited by a source of external current of the following form:

$$
\begin{aligned}
j(r, \phi, z, t) &= \begin{pmatrix} j_r \\ j_\phi \\ j_z \end{pmatrix} \equiv \begin{pmatrix} 0 \\ j_\phi \\ 0 \end{pmatrix}, \\
j_\phi(r, z, t) &= f(t)\delta(r - r_0)\delta(z - z_*),
\end{aligned}
\tag{1}
$$

where z_* is the coordinate of the source on the Oz axis, $z_* < 0$ (the source is in the air), z_* is small enough, $r_0 > 0$ is the source parameter.

Since the medium is assumed to be isotropic and the source does not depend on the angle ϕ, the Maxwell equations can be written in cylindrical coordinates, and the components of the electromagnetic field not depend on the angle ϕ. Taking into account the type of source (1), three of the six components of the electromagnetic field E_ϕ, H_r and H_z are non-zero (see, for example, [33, 34]). For the component E_ϕ the following differential equation can be obtained

$$
\varepsilon \frac{\partial^2 E_\phi}{\partial t^2} + \sigma \frac{\partial E_\phi}{\partial t} = \frac{1}{\mu}\left[\frac{\partial^2 E_\phi}{\partial z^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{1}{\partial r}\left(r E_\phi\right)\right)\right] - \frac{\partial j_\phi}{\partial t} \tag{2}
$$

At the initial moment of time there is no electromagnetic field

$$E_\phi|_{t<0} \equiv 0. \tag{3}$$

The tangential components of the electromagnetic field when passing through the interface between the media remain continuous, therefore, at the interface "ground-air" z_0 and the interface points z_n the gluing conditions

$$\big[E_\phi\big]_{z_n} = 0, \quad \left[\frac{\partial E_\phi}{\partial z}\right]_{z_n} = 0, \qquad n = \overline{0, M}. \tag{4}$$

are satisfied. The boundary condition

$$E_\phi|_{r=0} = 0, \tag{5}$$

holds, its rationale for which can be found in [35].

Let's assume that the following measurements have been made on the surface

$$E_\phi|_{z=0} = \xi(r,t). \tag{6}$$

We assume that the receivers recording electrograms are located along the r axis. Let us apply the Laplace and the Hankel transforms to the function $E_\phi(r,z,t)$:

$$u(\nu, z, p) = \int_0^\infty e^{-pt} \int_0^\infty r E_\phi(r,z,t) J_1(\nu r) dr dt, \tag{7}$$

where $p = \alpha + i2\pi \mathrm{f}$ is the Laplace transform parameter (α is the attenuation parameter, f is the time frequency (Hz)), $J_1(r)$ is the 1-st order Bessel function (see, for example, [36, 37]), ν is the Hankel transform parameter.

Let's introduce the following notation: $\xi(\nu,p)$ – image of the function $\xi(r,t)$, $f(p)$ – Laplace image for the function $f(t)$, $k(z) = \sqrt{\nu^2 + p^2\mu\epsilon_0\epsilon(z) + p\mu\sigma(z)}$ $(\mathrm{Re}\big\{k(z)\big\} > 0)$, in n-th layer $k_n = \sqrt{\nu^2 + p^2\mu\epsilon_0\epsilon_n + p\mu\sigma_n}$, in the air $k_0 = \sqrt{\nu^2 + p^2\mu\epsilon_0}$, and $g(\nu) = r_0 J_1(\nu r_0)$.

For $z \in \{(-\infty,\infty)\backslash\{z_0\}...\backslash\{z_M\}\}$ the function $u(\nu,z,p)$ satisfies the following differential equation:

$$u_{zz} - k^2(z)u = \mu p f(p) g(\nu) \delta(z - z_*), \tag{8}$$

and at the points $z = z_n$ $(n = \overline{0, M})$ the gluing conditions

$$[u_z]_{z_n} = 0, \quad [u]_{z_n} = 0, \tag{9}$$

hold, and additionally we must assume damping conditions

$$u|_{z\to\pm\infty} = 0. \tag{10}$$

Since $\delta(z - z_*)$ is present in the source (1), for $z \in \{(-\infty,\infty)\backslash\{z_*\}\backslash\{z_0\}...\backslash\{z_M\}\}$ we can consider the homogeneous differential equation

$$u_{zz} - k^2 u = 0, \tag{11}$$

with additional gluing conditions at the point z_*

$$[u_z]_{z_*} = \mu p f(p) g(\nu), \quad [u]_{z_*} = 0. \tag{12}$$

From (6) follows

$$u|_{z=0} = \xi(\nu, p). \tag{13}$$

Requirement $z_* \neq 0$ is mathematical because the product $\delta(z)w(z)$ has no a sense if the function w(z) is discontinuous at the point $z = 0$. Using the smallness of the value z_*, we can go to the limit $z_* \to 0$ and simplify the statement (8)-(12).

Using the condition of damping at $-\infty$ (10), the solution of differantial equation (11) in the interval $(-\infty, z_*)$, the gluing condition (12), the solution of differantial equation (11) in the interval $(z_*, 0)$, the gluing condition (9) at point $z = 0$, we can go to the limit $z_* \to 0$ and obtain the boundary condition

$$(u_z - k_0 u)|_{z=0} = \mu p f(p) g(\nu). \tag{14}$$

Thus, the function $u(\nu, z, p)$ can be found as solution of the differantial equation (11) for $z \in \{(0, \infty)\backslash\{z_0\}...\backslash\{z_M\}\}$, for wich the boundary condition (14), the condition of damping at ∞ (10), and the gluing condition (9) at the points $z = z_n$ $(n = \overline{1, M})$ hold. Here it is assumed that the electromagnetic properties of the layers $[z_{n-1}, z_n]$ $(n = \overline{1, M})$ and the underlying half-space $[z_M, \infty)$ are known.

Let us assume that the electromagnetic properties of layers from the first to the N-th are known, but below are not.

We could formulate inverse problem statement for reconstruction unkown electromagnetic constants ϵ_n and σ_n $(n = \overline{N+1, M})$ using additional information (13). But in this case we would need to solve the direct problem in the interval $z \in [0, z_M]$. We would like to use the knowledge that the electromagnetic properties of the first N layers are known and solve the direct problem in the interval $z \in [z_N, z_M]$. Reducing the interval over which the direct problem is solved significantly reduces the time for solving the direct problem and, consequently, the inverse problem, since solving the inverse problem using the optimization method is a multiple solution of the direct problem.

Therefore, we need to put the continuation problem: assuming that the $\epsilon(z)$ and $\sigma(z)$ $(z \in [0, z_N])$ are known piecewise constant functions, it is necessary to find expressions for values

$$u_z|_{z=z_N} \quad \text{and} \quad u|_{z=z_N}, \tag{15}$$

where the function $u(\nu, z, p)$ satisfies the differential equation (11) and boundary conditions (13) and (14).

This statement is the Cauchy problem for a second-order differential equation. This problem is well-posed. Thus, it is necessary to propose a stable algorithm for the numerical determination of the values (15).

3 Derivating the main relations

Let us consider the new functions $\alpha(z)$ and $\beta(z)$, which are connext with the function $u(\nu, z, p)$ by the following equality:

$$u_z = \alpha(z) u + \beta(z). \tag{16}$$

It should be noted here that since the constants ν and p are included as parameters in the formulation of (11), (9), (13) and (14), then the functions $\alpha(z)$ and $\beta(z)$ depend on them, but for brevity this dependence is not indicated.

Substituting the relation (16) into (11), (9) and (14), we obtain that the functions $\alpha(z)$ and $\beta(z)$ satisfy the following problems ($n=\overline{1,N-1}$):

$$\alpha'+\alpha^2=k^2,\quad \alpha(0)=k_0,\quad [\alpha]_{z_n}=0, \tag{17}$$

$$\beta'+\alpha\beta=0,\quad \beta(0)=\mu p f(p) g(\nu),\quad [\beta]_{z_n}=0. \tag{18}$$

For brevity, we denote: $\alpha^n=\alpha(z_n)$, $\beta^n=\beta(z_n)$, $u_n=u(\nu,z_n,p)$.

In each n-th layer, the solution to the Riccati differential equation (17) can be represented in the following forms:

$$\alpha(z)=k_n\frac{(\alpha^{n-1}+k_n)+(\alpha^{n-1}-k_n)e^{-2k_n(z-z_{n-1})}}{(\alpha^{n-1}+k_n)-(\alpha^{n-1}-k_n)e^{-2k_n(z-z_{n-1})}}, \tag{19}$$

$$\alpha(z)=k_n\frac{(\alpha^{n}+k_n)e^{2k_n(z-z_n)}+(\alpha^{n}-k_n)}{(\alpha^{n}+k_n)e^{2k_n(z-z_n)}-(\alpha^{n}-k_n)}. \tag{20}$$

From (19) and the gluing conditions (17) it follows that α^n can be defined "top to bottom" recursively:

$$\alpha^0=k_0,\quad \alpha^n=k_n\frac{(\alpha^{n-1}+k_n)+(\alpha^{n-1}-k_n)e^{-2k_nh_n}}{(\alpha^{n-1}+k_n)-(\alpha^{n-1}-k_n)e^{-2k_nh_n}},\quad n=\overline{1,N}, \tag{21}$$

and from (20) and the gluing conditions (17) it follows that α^n can be defined "bottom to top" recursively:

$$\alpha^{n-1}=k_n\frac{(\alpha^{n}+k_n)e^{-2k_nh_n}+(\alpha^{n}-k_n)}{(\alpha^{n}+k_n)e^{-2k_nh_n}-(\alpha^{n}-k_n)},\quad n=\overline{N,1}. \tag{22}$$

If in the relation (21) we express α^{n-1} in terms of α^n, then the relation (22) will be obtained, and vice versa, if in the relation (22) express α^n in terms of α^{n-1}, then the relation (21) will be obtained.

In each n-th layer, the solution to the differential equation from (18) can be represented in the following forms:

$$\beta(z)\;=\;\beta^{n-1}e^{-\int_{z_{n-1}}^{z}\alpha(x)dx}=\frac{2k_n\beta^{n-1}e^{-k_n(z-z_{n-1})}}{(\alpha^{n-1}+k_n)-(\alpha^{n-1}-k_n)e^{-2k_n(z-z_{n-1})}}, \tag{23}$$

$$\beta(z)\;=\;\beta^{n}e^{-\int_{z_{n}}^{z}\alpha(x)dx}=\frac{2k_n\beta^{n}e^{k_n(z-z_{n})}}{(\alpha^{n}+k_n)e^{2k_n(z-z_n)}-(\alpha^{n}-k_n)}, \tag{24}$$

From (23) and the gluing conditions (18) it follows that β^n can be determined "top to bottom" recurrently:

$$\beta^0=\mu p f(p) g(\nu),\quad \beta^n=\frac{2k_n\beta^{n-1}e^{-k_nh_n}}{(\alpha^{n-1}+k_n)-(\alpha^{n-1}-k_n)e^{-2k_nh_n}},\quad n=\overline{1,N}. \tag{25}$$

Note that in the expressions (19)-(25) there are only exponential functions whose real parts of the exponents are negative. This means that the calculation of the functions $\alpha(z)$ and $\beta(z)$ and recurrent recalculation will be done without the accumulation of rounding errors.

To obtain expressions that allow calculations without accumulating rounding errors, in each n-th layer the solution to the differential equation (16) can be presented in the following form:

$$u(\nu,z,p)=u_{n-1}\,e^{\int\limits_{z_{n-1}}^{z}\alpha(x)dx}+e^{\int\limits_{z_{n-1}}^{z}\alpha(x)dx}\int\limits_{z_{n-1}}^{z}\beta(y)\,e^{-\int\limits_{z_{n-1}}^{y}\alpha(x)dx}dy.$$

Using the representations (20) and (24) we obtain

$$\begin{aligned}u(\nu,z,p) &= u_{n-1}\,e^{-k_n(z-z_{n-1})}\frac{(\alpha^n+k_n)e^{2k_n(z-z_n)}-(\alpha^n-k_n)}{(\alpha^n+k_n)e^{-2k_nh_n}-(\alpha^n-k_n)}\\ &\quad+\beta^n e^{k_n(z_n-z)}\frac{e^{-2k_nh_n}-e^{2k_n(z-z_n)}}{(\alpha^n+k_n)e^{-2k_nh_n}-(\alpha^n-k_n)}.\end{aligned}\tag{26}$$

From (26) and the gluing conditions (9) it follows that u_n can be determined recurrently:

$$\begin{aligned}u_0 &= \xi(\nu,p),\\ u_n &= \frac{2k_n}{(\alpha^n+k_n)e^{-2k_nh_n}-(\alpha^n-k_n)}\left(u_{n-1}e^{-k_nh_n}-\beta^n\frac{1-e^{-2k_nh_n}}{2k_n}\right),\quad n=\overline{1,N}.\end{aligned}\tag{27}$$

The obtained relationships allow us to suggest the algorithm for solving the continuation problem, which will be presented in the next section. The denominator of the first factor in (27) does not vanish, since the Cauchy problem for a second-order differential equation for a layer has a limited unique solution, but it can be quite small, if the product $h_n\mathrm{Re}\{k_n\}$ is large.

4 Solving the continuation problem

First, calculate α^n and β^n ($n=\overline{1,N}$) with the help of the recurrent formulas (21) and (25).

Second, using known α^n and β^n ($n=\overline{1,N}$) calculate u^n ($n=\overline{1,N}$) with the help of the recurrent formula (27).

Thus, the following values

$$u|_{z=z_N}=u_N \quad\text{and}\quad u_z|_{z=z_N}=\alpha^N u_N+\beta^N \tag{28}$$

are obtained which means that the stated problem of continuation of the electromagnetic field in the frequency domain has been solved numerically. Note that all expressions for calculations are calculated stably, without accumulating rounding errors.

Knowing the quantities (28) allows us to formulate the direct problem statement:

$$u_{zz} - k^2 u = 0, \quad z \in [z_N, z_M], \tag{29}$$
$$u_z|_{z=z_N} = \alpha^N u_N + \beta^N, \tag{30}$$
$$(u_z + k_{M+1} u)|_{z=z_M} = 0, \tag{31}$$
$$[u_z]_{z_n} = 0, \quad [u]_{z_n} = 0, \quad n = \overline{N+1, M-1}, \tag{32}$$

and assume that additional information

$$u|_{z=z_N} = u_N \tag{33}$$

is known. The boundary condition (31) was obtained similarly to the condition (14).

The inverse problem (29)-(33) can be solved using a well-known approach (see, for example, [31]). Since the inverse problem is solved on a smaller interval $[z_N, z_M]$, the speed of determining the parameters ϵ_n and σ_n $(n = \overline{N+1, M})$ will be higher.

5 Conclusion

In this work, analytical expressions are obtained for solving the contimuation problem for the electromagnetic field in the frequency domain for horizontally layered media. The algorithm for the numerical solution of the problem is based on the layer-by-layer recalculation method. The resulting analytical expressions are presented in the form that allows calculations and layer-by-layer recalculation without the accumulation of rounding errors.

Acknowledgement

The work of the 3nd author was supported within the framework of the state assignment of Sobolev Institute of Mathematics SB RAS (project FWNF-2022-0009).

References

[1] Tikhonov A. N., Shakhsuvarov D. N., *A method for modeling TEM fields in layered media*, Izv. AN SSSR, Ser. Geofiz., 3. (1956), 251-254.

[2] Dmitriev V. I., *A universal method for inversion of the electromagnetic field in layered media*, Vychislitelnye Metody i Programmirovaniye, MSU, 10. (1968), 55-65.

[3] Dmitriev V. I., Fedorova E. A., *Numerical modeling of electromagnetic fields in layered media*, Vychislitelnye Metody i Programmirovaniye, MSU, 32 (1980), 150-183.

[4] Gel'fand I. M., Lokutsievskii O. V., *The "sweep" method for solving difference equations*, In book: Godunov S. K., Ryaben'kii V. S., Introduction to the theory of difference schemes. Moskow, Fizmatgiz, 1962, 283-309.

[5] Akkuratov G.V., Dmitriev V.I., *A method for inversion of stationary elastic wavefields in layered media*, Chislennye Metody v Geofizike, Bull. Moscow Univ., (1979) 3-12.

[6] Akkuratov, G.V., Dmitriev, V.I., 1984. A method for inversion of stationary elastic wave fields in layered media. Zhurnal Vychislitelnoi Matematiki i Matematicheskoi Fiziki, 24, 2 (1984), 272-286. (In Russian)

[7] Pavlov V. M., *A convienent technique for calculating synthetic seismograms in layered half-space*, Proceedings of the International Conference "Problems of Geocosmos", St. Peteburg, 03-08 June 2002, 320–323.

[8] *Fatianov A. G., Mikhailenko B.G., A method for inversion of non-stationary wave fields in non-elastic layered media*, Dokl. Akad. Nauk SSSR, 301. 4 (1988), 834-839.

[9] Fatianov A. G., Semi-analytical solutions to forward dynamic problems for layered media. Dokl. Akad. Nauk SSSR 310. 2 (1990), 323-327.

[10] Fatianov, A.G., Non-stationary wave fields in inhomogeneous anisotropic media with absorption. Preprint 857, Computing Center SB RAS, Novosibirsk, 1989. (In Russian)

[11] Karchevsky A. L. *A numerical solution to a system of elasticity equations for layered anisotropic media*, Russian Geology and Geophysics, 46. 3 (2005), 339-351.

[12] Karchevsky A. L. *Direct dynamical problem of seismics for horizontaly stratified media*, Siberian Electronic Math. Reports, 2. (2005), 23-61. http://semr.math.nsc.ru/v2/p23-61.pdf

[13] Karchevsky A. L. *The analytical formulas for the gradient of the residual functional for the coefficient inverse problem for the elasticity system*, Journal of Inverse and Ill-Posed Problems, 11. 6 (2003), 619-629. https://doi.org/10.1515/156939403322759679

[14] Kurpinar E., Karchevsky A. L., *Optimization inversion of seismic data from layered media: an algorithm for gradient*. Russian Geology and Geophysics, 46. 4 (2005), 439-447.

[15] Karchevsky A. L., *A frequency-domain analytical solution of Maxwell's equations for layered anisotropic media*. Russian Geology and Geophysics, 48. 8 (2007), 689-695. https://doi.org/10.1016/j.rgg.2006.08.005

[16] Karchevsky A. L., Rysbayuly B. R., *Analitical expressions for a solution of convective heat and moisture transfer equations in the frequency domain for layered media*, Eurasian Journal of Mathematical and Computer Applications, 3. 4 (2015), 55-67. http://www.ejmca.enu.kz/images/stories/3vypusk/karchevsky_ejmca.pdf

[17] Karchevsky A. L., *Analytical Solutions to the Differential Equation of Transverse Vibrations of a Piecewise Homogeneous Beam in the Frequency Domain for the Boundary Conditions of Various Types*, Journal of Applied and Industrial Mathematics, 14. 4 (2020), 52-69. https://doi.org/10.1134/S1990478920040043

[18] Somersalo E., Cheney M., Isaacson D., Isaacson E., *Layer stripping: a direct numerical method for impedance imaging*. Inverse Problems, 7. (1991), 899-926. https://doi.org/10.1088/0266-5611/7/6/011

[19] Gribov A. F., Zhidkov E. N., Krasnov I. K. *On the numerical solution of the inverse problem of heat conduction with radiation*, Mathematical modeling and numerical methods, 1. (2019), 43-53.

[20] Abgarian K. K., Noskov R. G., Reviznikov D.L., *The inverse coefficient problem of heat transfer in layered nanostructures*, Izvestiya vuzov. Materialy elektronnoi tekhniki, 20. 3 (2017), 213-219. https://doi.org/10.17073/1609-3577-2017-3-213-219

[21] Popov K. V., Tikhonravov A. V., *The inverse problem in optics of stratified media with discontinuous parameters*, Inverse Problems, 13. (1997), 801-814. https://doi.org/10.1088/0266-5611/13/3/015

[22] Habashy T. M., Mittra R., *Review of some inverse methods in electromagnetics*, Journal of the Optical Society of America A, 4. 1 (1987), 281-291 (1987) https://doi.org/10.1364/JOSAA.4.000281

[23] Durdiev U. D., *Numerical method for determining the dependence of the dielectric permittivity on the frequency in the equation of electrodynamics with memory*, Siberian Electronic Mathematical Reports, 17. (2020), 179-189. https://doi.org/10.33048/semi.2020.17.013

[24] Bozorov Z. R. *Numerical determining a memory function of a horizontallystratified elastic medium with aftereffect*, Eurasian journal of mathematical and computer applications, 8. 2 (2020), 4-16. https://doi.org/10.32523/2306-6172-2020-8-2-4-16

[25] Iskakov K. T., Boranbaev S. A., Uzakkyzy N. *Wavelet processing and filtering of the radargram trace*, Eurasian J. Math. Comput. Appl., 5. 4 (2017) 43-54. https://ejmca.enu.kz/assets/files/5-4-4.pdf

[26] Iskakov K., Sagindykov K., Mukhambetkaliyev K., Kalmenov K., Seitkhanova A. *Study of Pavement Anomalies Using GPR of OKO-2 series*, Material and Mechanical Engineering Technology, 4. (2023) 42-46. https://doi.org/10.52209/2706-977X_2023_4_42

[27] Iskakov K., Boranbaev S., Alimbayeva Zh., Isin B. *Experimental data of research using ground-penetrating radar Zond-12c and interpretation of georadarograms*, Acta Phys. Pol. A, 130. 1 (2016), 322-324. https://doi.org10.12693/APhysPolA.130.322

[28] Karchevsky A. L., *Numerical Solution to the One-Dimensional Inverse Problem for an Elastic System*, Doklady Akademii Nauk, 375. 8 (2000), 235-238.

[29] Karchevsky A. L., Fatianov A. G., *Numerical solution of the inverse problem for a system of elasticity with the aftereffect for a vertically inhomogeneous medium*, Sib. Zh. Vychisl. Mat., 4. 3 (2001), 259-268.

[30] Karchevsky A. L., *Numerical reconstruction of medium parameters of member of thin anisotropic layers*, Journal of Inverse and Ill-Posed Problems, 12. 5 (2004), 519-634. https://doi.org/10.1515/1569394042531332

[31] Karchevsky A. L., *Simultaneous reconstruction of permittivity and conductivity*, Journal of Inverse and Ill-Posed Problems, 17. 4 (2009), 385-402. https://doi.org/10.1515/JIIP.2009.026

[32] Duchkov A. A., Karchevsky A. L., *Determination of Terrestrial Heat Flow from Temperature Measurements in Bottom Sediments*, Journal of Applied and Industrial Mathematics, 7. 4 (2013), 480-502. https://doi.org/10.1134/S1990478913040042

[33] Romanov V.G., Karchevsky A.L., *Determination of permittivity and conductivity of medium in a vicinity of a well having complex profile*, Eurasian Journal of Mathematical and Computer Applications, 6. 4 (2018), 64-74. https://doi.org/10.32523/2306-6172-2018-6-4-62-72

[34] Romanov V. G., Kabanikhin S. I. Inverse Problems for Maxwell's Equations. VSP, Utrecht, 1994.

[35] Romanov V. G., Kabanikhin S. I., Shishlenin M. A. *Investigation of mathematical model of electromagnetic probe in axially symmetrical borehole*. Siberian Electronic Mathematical Reports, 7. (2010), C.307-C.321. http://semr.math.nsc.ru/v7/1-394.pdf

[36] Gradshtein I. S., Ryzhik I. M. Tables of integrals, sums, series and products. 4th edition, Moscow, Fizmatgiz, 1963. (In Russian).

[37] Janke E., Emde F. Lösch F. Tafeln Höherer Funktionen. Stuttgart, B.G. Verlagsgesellschaft, 1960.

[38] Tikhonov A. N. Collection of scientific works. In 10 volumes, Moscow, Nauka, 2009. (In Russian)

[39] Tikhonov A. N., Arsenin V. Ya., Methods for solving ill-posed problems. Moskow, Nauka, 1979. (In Russian)

[40] Lavrent'ev M. M., Romanov V. G., Vasil'ev V. G., Multidimensional inverse problems for differential equations. Novosibirsk, Nauka, 1969. (In Russian)

[41] Lavrent'ev M. M., Romanov V. G., Shishatskii S.P., Some problems of mathematical physics and analysis. Novosibirsk, Nauka, 1980. (In Russian)

[42] Vasin V. V., Ageev A.L., Ill-posed problems with a priori information. Ekaterinburg, Nauka, 1993. (In Russian)

[43] Vasin V. V., Eremin I. I., Operators and iteration processes of the Feyer type. Theory and applications. Moscow-Izhevsk, Institute of Computer Technologies, Research Center "Regular and Chaotic Dynamics", 2005.

[44] Yagola A. G., Yanfei V., Stepanova I. E., Taranenko V.N., Inverse problems and methods for their solution. Applications in geophysics. Moskow, BINOM, Laboratoriya znanii, 2014.

[45] Sibiryakov N. E., Kochkin D. Yu., Kabov O. A., Karchevsky A.L., *Determining the Heat Flux Density in the Area of a Contact Line during the Evaporation of Liquid into a Bubble*, Journal of Applied and Industrial Mathematics, 17. 3 (2023), 628-639. https://doi.org/10.1134/S199047892303016X

[46] Belonosov A., Shishlenin M., *Regularization Methods of the Continuation Problem for the Parabolic Equation*, In: Dimov I., Farago I., Vulkov L. (Editors), Numerical Analysis and Its Applications - 6th International Conference, NAA 2016, Revised Selected Papers (p. 220-226). Springer-Verlag GmbH and Co. KG. https://doi.org/10.1007/978-3-319-57099-0_22

[47] Karchevsky A. L., *Reformulation of an inverse problem statement that reduces computational costs*, Eurasian Journal of Mathematical and Computer Applications, 1. 2 (2013), 4-20. https://ejmca.enu.kz/assets/files/1-2-1.pdf

[48] Karchevsky A. L., Marchuk I. V., Kabov O. A., *Calculation of the heat flux near the liquid-gas-solid contact line*, Applied Mathematical Modelling, 40. 2 (2016), 1029-1037. https://doi.org/10.1016/j.apm.2015.06.018

[49] Cheverda V. V., Marchuk I. V., Karchevsky A. L., Orlik E. V., Kabov O. A., *Experimental investigation of heat transfer in a rivulet on the inclined foil*, Thermophysics and Aeromechanics, 23. 3 (2016), 415-420. https://doi.org/10.1134/S0869864316030112

[50] Cheverda V. V., Karchevsky A. L., Marchuk I. V., Kabov O. A., *Heat flux density in the region of droplet contact line on a horizontal surface of a thin heated foil*, Thermophysics and Aeromechanics, 24. 5 (2017), 803-806. https://doi.org/10.1134/S086986431705016X

[51] Karchevsky A. L., *Development of the heated thin foil technique for investigating nonstationary transfer processes*, Interfacial Phenomena and Heat Transfer, 6. 3 (2018), 179-185. https://doi.org/10.1615/InterfacPhenomHeatTransfer.2018028949

[52] Karchevsky A. L., *Numerical solving the heat equation with data on a time-like boundary for the heated thin foil technique*, Eurasian Journal of Mathematical and Computer Application, 8. 4 (2020), 4-14. https://doi.org/10.32523/2306-6172-2020-8-4-4-14

[53] Yaparova N. M., *On various approaches to solving inverse boundary value problems of thermal diagnostics*, Vestn. SUSU, Series “Math., Mech., Phys.”, 7. (2012), 60-67.

[54] Yaparova N. M., *Numerical modeling of solutions to the inverse boundary value problem of heat conduction*, Vestn. SUSU, series “Math. modeling and programming”, 6. 3 (2013), 112-124.

[55] Solodusha S. V., Yaparova N. M., *Numerical solving an inverse boundary value problem of heat conduction using Volterra equations of the first kind*, Numer. Analys. Appl., 8. (2015), 267-274. https://doi.org/10.1134/S1995423915030076

Orman I. M.,
Manash Kozybayev North Kazakhstan university
st. Pushkina, 86, 150000 Petropavlovsk, Kazakhstan,
Email: Indira.malikovna@mail.ru,

Kurmashev I. G.,
Manash Kozybayev North Kazakhstan university
st. Pushkina, 86, 150000 Petropavlovsk, Kazakhstan,
Email: ikurmashev@ku.edu.kz,

Karchevsky A. L.,
Sobolev Insritute of Mathematics SB RAS,
pr. Akad. Koptyuga, 630090 Novosibirsk, Russia,
Email: karchevs@math.nsc.ru.

Received 13.02.2024, revised 15.03.2024 Accepted 20.03.2024