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BOUNDARY CONTROL PROBLEM FOR A PARABOLIC EQUATION WITH INVOLUTION

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Abstract In this paper, we consider a boundary control problem for a parabolic equation with involution in a bounded one-dimensional domain. On the part of the border of the considered domain, the value of the solution with control function is given. Restrictions on the control are given in such a way that the average value of the solution in the considered domain gets a given value. The problem given by the method of separation of variables is reduced to the Volterra integral equation of the first kind. The existence of the control function was proved by the Laplace transform method.

Key words: parabolic equation, boundary problem, Volterra integral equation, control function, Laplace transform, involution.

AMS Mathematics Subject Classification: 35K05, 35K15.

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1 Introduction

In this article, we consider the following parabolic equation with involution in the domain $\Omega_T := (0,\pi) \times (0,\infty)$

$$u_t(x,t) - u_{xx}(x,t) + \varepsilon\, u_{xx}(\pi - x, t) \;=\; 0, \quad (x,t) \in \Omega_T, \tag{1}$$

with Neumann boundary conditions

$$u_x(0,t) \;=\; -\nu(t), \quad u_x(\pi,t) \;=\; 0, \quad t \geq 0, \tag{2}$$

and initial condition

$$u(x,0) \;=\; 0, \quad 0 \leq x \leq \pi, \tag{3}$$

where ε is a nonzero real number such that $|\varepsilon| < 1$, and $\nu(t)$ is the control function. If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0) = 0$ and $|\nu(t)| \leq 1$ on the half-line $t \geq 0$, then we call it an admissible control.

We will prove later in Section 3 that the function ν belongs to the class $W_2^1(\mathbb{R}_+)$.

Differential equations with modified arguments are equations in which the unknown function and its derivatives are evaluated with modifications of time or space variables; such equations are called, in general, functional differential equations. Among such equations, one can single out, equations with involutions [1].

Definition 1. *([2, 3]) A function $f(x) \not\equiv x$ maps bijectively a set of real numbers Ω, such that*

$$f(f(x)) \;=\; x \quad or \quad f^{-1}(x) \;=\; f(x),$$

is called an involution on Ω.

Due to the widespread use of partial differential equations in physics and technology, there is always a great interest in the study of boundary value control problems. For this purpose, various boundary problems for parabolic and pseudo-parabolic equations have been widely studied by many researchers.

It can be seen that equation (1) for $\varepsilon = 0$ is a classical parabolic equation. If $\varepsilon \neq 0$, equation (1) relates the values of the second derivatives at two different points and becomes a nonlocal equation. It is known that boundary control problems for the parabolic equation in the case $\varepsilon = 0$ were studied in details in work [4].

Assume that the function $\rho(x) \in W_2^2(0, \pi)$ satisfies the conditions

$$\rho'(x) \leq 0, \quad \rho''(x) \geq 0, \quad \int_0^\pi \rho(x)\, dx = 1. \tag{4}$$

Let

$$\rho(x) = \sum_{k=0}^{\infty} \rho_k \cos kx, \quad x \in (0, \pi),$$

where

$$\rho_0 = \frac{1}{\sqrt{\pi}} \int_0^\pi \rho(x)\, dx = \frac{1}{\sqrt{\pi}}, \quad \rho_k = \sqrt{\frac{2}{\pi}} \int_0^\pi \rho(x) \cos kx\, dx, \quad k = 1, 2, ... \tag{5}$$

We now consider the following control problem.

Control Problem. For the given function $\phi(t)$ Problem consists of looking for the admissible control $\nu(t)$ such that the solution $u(x,t)$ of the initial-boundary problem (1)-(3) exists and for all $t \geq 0$ satisfies the equation

$$\int_0^\pi \rho(x)\, u(x,t)\, dx = \phi(t), \quad t \geq 0. \tag{6}$$

The optimal control problem for the parabolic type equations was studied by Fattorini and Friedman [5, 6]. Control problems for the infinite-dimensional case were studied by Egorov [7], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The boundary control problem for a parabolic equation with a piecewise smooth boundary in an $n-$dimensional domain was studied in [8] and an estimate for the minimum time required to reach a given average temperature was found. In [9], mathematical models of thermocontrol processes for the parabolic equation are considered. Control problems for the heat transfer equation in the three-dimensional domain are studied in [10].

Control problems for parabolic equations in bounded one and two-dimensional domains are studied in works [11, 12, 13, 14]. In these articles, an estimate was found for the minimum time required to heat a bounded domain to an estimate average temperature. The existence of control function is proved by Laplace transform method.

Basic information on optimal control problems is given in detail in monographs by Lions and Fursikov [15, 16]. General numerical optimization and optimal control for second-order parabolic equations have been studied in many publications such as [17]. Practical applications of optimal control problems for equations of parabolic type were presented in [18]. The control problems for the pseudoparabolic equation in the bounded domain were studied in works [19, 20] and the existence of an admissible control function was proved using the Laplace transform method.

It is known that in recent years, due to the increasing interest in physics and mathematics, the boundary problems related to heat diffusion equations related to involution were widely studied. In [21], a boundary value problem for the heat equation associated with involution in a one-dimensional domain is studied. Many boundary value problems for parabolic type equations with involution were studied in works [22, 23].

Finding the control function $\nu(t)$ from the additional integral condition (6) with the solution of the mixed problem (1)-(3) can be also considered as an inverse problem. In [25], the inverse problem for the non-homogeneous parabolic type equation was considered and the uniqueness theorem was proved. Inverse problems for integro-differential equations of electrodynamics with dispersion and viscoelasticity equations were considered in [26].

In [27], the inverse problem of determining the time-dependent reaction diffusion coefficient in the Cauchy problem for the fractional time diffusion equation was studied. One-dimensional inverse problems for systems of isotropic, anisotropic orthorombic, and anisotropic hexagonal (transversally isotropic) elasticmedia were studied in [28].

In this work, the boundary control problem for the parabolic equation with involution is considered. The boundary control problem studied in this work is reduced to the Volterra integral equation of the first kind by the Fourier method (Section 2). In Section 3, the existence of a solution to the integral equation is proved using the Laplace transform method.

2 Volterra integral equation

In this section, we consider how the given control problem can be reduced to a Volterra integral equation of the first kind.

We now consider the spectral problem

$$X''(x) - \varepsilon\, X''(\pi - x) + \lambda\, X(x) \ = \ 0, \quad 0 < x < \pi,$$

$$X'(0) = X'(\pi) = 0, \quad 0 \le x \le \pi,$$

where $|\varepsilon| < 1$, $\varepsilon \in \mathbb{R} \setminus \{0\}$. It is proved in [23, 24] that expressing the solution of spectral problem in terms of the sum of even and odd functions, one finds the following eigenvalues:

$$\lambda_{2k+1} = (1+\varepsilon)\,(2k+1)^2, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

$$\lambda_{2k} = (1-\varepsilon)\,4\,k^2, \quad k \in \mathbb{N}_0,$$

and we have the following eigenfunctions

$$X_0 = \frac{1}{\sqrt{\pi}}, \quad X_{2k} = \sqrt{\frac{2}{\pi}} \cos 2kx, \quad k \in \mathbb{N},$$

and

$$X_{2k+1} = \sqrt{\frac{2}{\pi}} \cos(2k+1)x, \quad k \in \mathbb{N}_0.$$

For an arbitrary Banach space B and for $T > 0$ by the symbol $C([0,T] \to B)$ we denote the Banach space of all continuous maps $u : [0,T] \to B$ with the norm

$$\|u\| = \max_{0 \le t \le T} |u(t)|.$$

By symbol $\widetilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ formed by functions, whose trace on $\partial\Omega$ is equal to zero. Note that due to the closure $\widetilde{W}_2^1(\Omega)$ the sum of a series of functions from $\widetilde{W}_2^1(\Omega)$, converging in metric $W_2^1(\Omega)$ also belongs to $\widetilde{W}_2^1(\Omega)$, where $\Omega := \{x : 0 < x < \pi\}$.

Definition 2. *By the solution of the problem* (1) - (3) *we mean a function* $u(x,t)$, *represented in the form*

$$u(x,t) = \nu(t) \frac{(\pi - x)^2}{2\,\pi} - w(x,t), \tag{7}$$

where the function $w(x,t)$ *is a generalized solution from the class* $C([0,T] \to \widetilde{W}_2^1(\Omega))$ *of the following problem:*

$$w_t(x,t) - w_{xx}(x,t) + \varepsilon\, w_{xx}(\pi - x, t) \;=\; \frac{(\pi - x)^2}{2\,\pi}\, \nu'(t) + \frac{\varepsilon - 1}{\pi}\, \nu(t),$$

with homogeneous initial and boundary conditions

$$w_x(0,t) = w_x(\pi,t) = 0, \quad w(x,0) = 0.$$

Thus, we obtain (see [29])

$$\begin{aligned} w(x,t) \;&=\; \frac{\pi\sqrt{\pi}}{6}\, \nu(t) + \frac{\varepsilon - 1}{\sqrt{\pi}} \int_0^t \nu(s)\, ds + \\ &+ \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \left(\int_0^t e^{-\lambda_{2k+1}(t-s)}\, \nu'(s)\, ds \right) \cos(2k+1)x + \\ &+ \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{1}{4k^2} \left(\int_0^t e^{-\lambda_{2k}(t-s)}\, \nu'(s)\, ds \right) \cos 2kx. \end{aligned} \tag{8}$$

Note that the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ is a subset of the class $W_2^{1,0}(\Omega_T)$, which was considered in monograph [30] for defining a solution to the problem homogeneous boundary conditions (see the corresponding uniqueness theorem in Ch. III, Theorem 3.2, pp. 173-176). Therefore, the above introduced generalized solution is also a generalized solution in the sense of [30]. However, unlike a solution from the class $W_2^{1,0}(\Omega_T)$, which is guaranteed to have a trace for almost everywhere $t \in [0,T]$, a solution from a class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ continuously depends on $t \in [0,T]$ in the metric $L_2(\Omega)$.

Proposition 2.1. *Let $\nu \in W_2^1(\mathbb{R}_+)$, $\nu(0) = 0$ and $|\varepsilon| < 1$. Then the function*

$$u(x,t) = \frac{1-\varepsilon}{\sqrt{\pi}} \int\limits_0^t \nu(s)\,ds + (1+\varepsilon)\sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \Big(\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\,\nu(s)\,ds \Big) \cos(2k+1)x +$$

$$+(1-\varepsilon)\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \Big(\int\limits_0^t e^{-\lambda_{2k}(t-s)}\,\nu(s)\,ds \Big) \cos 2kx, \tag{9}$$

is the solution of the initial-boundary value problem (1)-(3).

Proof. Using (7) and (8), we rewrite the solution of the problem (1)-(3) in the form

$$u(x,t) = \nu(t)\,\frac{(\pi-x)^2}{2\,\pi} - \frac{\pi\sqrt{\pi}}{6}\,\nu(t) + \frac{1-\varepsilon}{\sqrt{\pi}} \int\limits_0^t \nu(s)\,ds -$$

$$-\sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \Big(\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\,\nu'(s)\,ds \Big) \cos(2k+1)x -$$

$$-\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{1}{4k^2} \Big(\int\limits_0^t e^{-\lambda_{2k}(t-s)}\,\nu'(s)\,ds \Big) \cos 2kx.$$

We will prove that function $w(x,t)$ represented by the indicated Fourier series, belongs to the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$. It suffices to prove that the gradient of this function, taken with respect to $x \in \Omega$, continuously depends on $t \in [0,T]$ on the norm of the space $L_2(\Omega)$. According to Parseval's equality, the norm of this gradient is equal to

$$\|w_x(\cdot,t)\|^2_{L_2(\Omega)} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \Big(\int\limits_0^t e^{-\lambda_{2k+1}(t-s)}\,\nu'(s)\,ds \Big)^2 +$$

$$+\sum_{k=1}^{\infty} \frac{1}{4k^2} \Big(\int\limits_0^t e^{-\lambda_{2k}(t-s)}\,\nu'(s)\,ds \Big)^2 \le C\,\|\nu'\|^2_{L_2(\mathbb{R}_+)} \sum_{k=1}^{\infty} \frac{1}{k^4} = C_1\,\|\nu'\|^2_{L_2(\mathbb{R}_+)}.$$

The fact that the function $w(x,t)$ is a generalized solution in the sense of the integral identity (3.5) of monograph [30] immediately follows from Parseval's equality. $\square$

Using the condition (6) and the solution (9), we can write

$$\phi(t) = \int_0^\pi \rho(x)\, u(x,t)\, dx = \frac{1-\varepsilon}{\sqrt{\pi}} \int_0^\pi \rho(x)\, dx \int_0^t \nu(s)\, ds +$$

$$+(1+\varepsilon)\sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \left(\int_0^t e^{-\lambda_{2k+1}(t-s)}\, \nu(s)\, ds \right) \int_0^\pi \rho(x)\, \cos(2k+1)x\, dx +$$

$$+(1-\varepsilon)\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \left(\int_0^t e^{-\lambda_{2k}(t-s)}\, \nu(s)\, ds \right) \int_0^\pi \rho(x)\, \cos 2kx\, dx.$$

By (4) and (5), we get

$$\phi(t) = \frac{1-\varepsilon}{\sqrt{\pi}} \int_0^t \nu(s)\, ds + (1+\varepsilon) \sum_{k=0}^{\infty} \rho_{2k+1} \int_0^t e^{-\lambda_{2k+1}(t-s)}\, \nu(s)\, ds +$$

$$+(1-\varepsilon) \sum_{k=1}^{\infty} \rho_{2k} \int_0^t e^{-\lambda_{2k}(t-s)}\, \nu(s)\, ds.$$

Set

$$K(t) = \frac{1-\varepsilon}{\sqrt{\pi}} + (1+\varepsilon) \sum_{k=0}^{\infty} \rho_{2k+1}\, e^{-\lambda_{2k+1}\, t} +$$

$$+(1-\varepsilon) \sum_{k=1}^{\infty} \rho_{2k}\, e^{-\lambda_{2k}\, t}, \quad t > 0, \tag{10}$$

where ρ_k is defined by (5).

Thus, we have the following Volterra integral equation of the first kind

$$\int_0^t K(t-s)\, \nu(s) ds = \phi(t), \quad t > 0. \tag{11}$$

3 Main result

In this section, we will consider on the existence of the control function.

Assume that $M > 0$ is a constant. Then we denote by $W(M)$ the set of functions $\phi \in W_2^2(-\infty, +\infty)$, $\phi(t) = 0$ for all $t \le 0$ which satisfy the condition

$$\|\phi\|_{W_2^2(R_+)} \le M.$$

We present the following main theorem.

Theorem 3.1. *There exists $M > 0$ such that for any function $\phi \in W(M)$ the solution $\nu(t)$ of the Voltera integral equation (11) exists and it satisfies condition $|\nu(t)| \le 1$.*

Remark 1. *The uniqueness of the solution of Volterra integral equation* (11) *follows from Titchmarsh's theorem (see, e.g., [31], Chapter VI, Section 5).*

Lemma 3.1. *[11] Let $g(x) \geq 0$ and $g'(x) \leq 0$ on $x \in [0, \infty)$. Then the following inequality holds:*

$$\int_0^{n\pi} g(x) \sin x \, dx \geq 0, \quad n = 1, 2, ...$$

Lemma 3.2. *The following estimate is valid:*

$$0 \leq \rho_k \leq \frac{C}{k^2}, \quad k = 1, 2, ...,$$

where ρ_k is defined by (5).

Proof. From (5), we write

$$\rho_k = \sqrt{\frac{2}{\pi}} \int_0^{\pi} \rho(x) \cos kx \, dx = \frac{1}{k} \sqrt{\frac{2}{\pi}} \rho(x) \sin kx \Big|_{x=0}^{x=\pi} -$$

$$- \frac{1}{k} \sqrt{\frac{2}{\pi}} \int_0^{\pi} \rho'(x) \sin kx \, dx = -\frac{1}{k} \sqrt{\frac{2}{\pi}} \int_0^{\pi} \rho'(x) \sin kx \, dx. \qquad (12)$$

By condition (4) and Lemma 3.1 we obtain $\rho_k \geq 0$. Then, from (12) we can write

$$\rho_k = -\frac{1}{k} \sqrt{\frac{2}{\pi}} \int_0^{\pi} \rho'(x) \sin kx \, dx = \frac{1}{k^2} \sqrt{\frac{2}{\pi}} \rho'(x) \cos kx \Big|_{x=0}^{x=\pi} -$$

$$-\frac{1}{k^2} \sqrt{\frac{2}{\pi}} \int_0^{\pi} \rho''(x) \cos kx \, dx = \frac{1}{k^2} \sqrt{\frac{2}{\pi}} \Big(\rho'(\pi) (-1)^k - \rho'(0) \Big) + \frac{o(1)}{k^2},$$

where $\rho'(\pi) (-1)^k - \rho'(0) \geq 0$.

Then we obtain

$$0 \leq \rho_k \leq \frac{C}{k^2}.$$

□

Proposition 3.1. *Assume that ε is a nonzero real number such that $|\varepsilon| < 1$. Then, the kernel $K(t)$ of the integral equation* (11) *is continuous on $t \geq 0$.*

Proof. By (10) and Lemma 3.2, we can write the following estimate for any $|\varepsilon| < 1$:

$$0 < K(t) \;=\; \frac{1-\varepsilon}{\sqrt{\pi}} + (1+\varepsilon)\sum_{k=0}^{\infty} \rho_{2k+1}\, e^{-\lambda_{2k+1}\, t} +$$

$$+(1-\varepsilon)\sum_{k=1}^{\infty} \rho_{2k}\, e^{-\lambda_{2k}\, t} \le \frac{2}{\sqrt{\pi}} + C_\varepsilon \sum_{k=1}^{\infty} \frac{1}{k^2},$$

where C_ε is a constant only depending on ε. □

We find the solution of the Volterra integral equation (11) using the method of Laplace transform. We know that

$$\widetilde{\nu}(p) = \int_0^{\infty} e^{-pt}\, \nu(t)\, dt.$$

Then using the Laplace transform we get the following equation

$$\widetilde{\phi}(p) = \int_0^{\infty} e^{-pt}\, dt \int_0^{t} K(t-s)\, \nu(s) ds = \widetilde{K}(p)\, \widetilde{\nu}(p).$$

Thus, we obtain

$$\widetilde{\nu}(p) = \frac{\widetilde{\phi}(p)}{\widetilde{K}(p)}, \quad \text{where } \; p = \xi + i\,\tau, \quad \xi > 0, \quad \tau \in \mathbb{R},$$

and we can write the function $\nu(t)$ as follows:

$$\nu(t) = \frac{1}{2\pi i} \int_{\xi - i\,\infty}^{\xi + i\,\infty} \frac{\widetilde{\phi}(p)}{\widetilde{K}(p)}\, e^{pt} dp = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(\xi + i\,\tau)}{\widetilde{K}(\xi + i\,\tau)}\, e^{(\xi + i\,\tau)t}\, d\tau. \tag{13}$$

Proposition 3.2. *The following estimate*

$$|\widetilde{K}(\xi + i\,\tau)| \ge \frac{C_\xi}{\sqrt{1+\tau^2}}, \quad \xi > 0, \quad \tau \in \mathbb{R},$$

is valid, where $C_\xi > 0$ is a constant only depending on ξ.

Proof. Set

$$\beta_{2k+1} = (1+\varepsilon)\, \rho_{2k+1}, \quad k \in \mathbb{N}_0,$$

and

$$\beta_{2k} = (1-\varepsilon)\, \rho_{2k}, \quad k \in \mathbb{N}.$$

Using the Laplace transform, we may write

$$\widetilde{K}(p)=\int\limits_0^\infty K(t)\,e^{-pt}\,dt=\frac{1}{\sqrt{\pi}}\frac{1-\varepsilon}{p}+\sum_{k=0}^\infty\frac{\beta_{2k+1}}{p+\lambda_{2k+1}}+\sum_{k=1}^\infty\frac{\beta_{2k}}{p+\lambda_{2k}},$$

where $K(t)$ is defined by (10) and

$$\begin{aligned}\widetilde{K}(\xi+i\,\tau)\ &=\ \frac{1}{\sqrt{\pi}}\frac{1-\varepsilon}{\xi+i\,\tau}+\\ &+\sum_{k=0}^\infty\frac{\beta_{2k+1}}{\xi+\lambda_{2k+1}+i\,\tau}+\sum_{k=1}^\infty\frac{\beta_{2k}}{\xi+\lambda_{2k}+i\,\tau}=\\ &=\frac{1}{\sqrt{\pi}}\frac{(1-\varepsilon)\,(\xi-i\,\tau)}{\xi^2+\tau^2}+\sum_{k=0}^\infty\frac{\beta_{2k+1}\,(\xi+\lambda_{2k+1})}{(\xi+\lambda_{2k+1})^2+\tau^2}-\\ &-i\,\tau\sum_{k=0}^\infty\frac{\beta_{2k+1}}{(\xi+\lambda_{2k+1})^2+\tau^2}+\sum_{k=1}^\infty\frac{\beta_{2k}\,(\xi+\lambda_{2k})}{(\xi+\lambda_{2k})^2+\tau^2}-\\ &-i\,\tau\sum_{k=1}^\infty\frac{\beta_{2k}}{(\xi+\lambda_{2k})^2+\tau^2}=\mathrm{Re}\widetilde{K}(\xi+i\,\tau)+i\,\mathrm{Im}\widetilde{K}(\xi+i\,\tau),\end{aligned}$$

where $\mathrm{Re}\widetilde{K}(\xi+i\,\tau)$ and $\mathrm{Im}\widetilde{K}(\xi+i\,\tau)$ are defined as follows:

$$\mathrm{Re}\widetilde{K}(\xi+i\,\tau)=\frac{1}{\sqrt{\pi}}\frac{(1-\varepsilon)\,\xi}{\xi^2+\tau^2}+\sum_{k=0}^\infty\frac{\beta_{2k+1}\,(\xi+\lambda_{2k+1})}{(\xi+\lambda_{2k+1})^2+\tau^2}+$$

$$+\sum_{k=1}^\infty\frac{\beta_{2k}\,(\xi+\lambda_{2k})}{(\xi+\lambda_{2k})^2+\tau^2},$$

and

$$\mathrm{Im}\widetilde{K}(\xi+i\,\tau)=-\frac{1}{\sqrt{\pi}}\frac{(1-\varepsilon)\,\tau}{\xi^2+\tau^2}-\tau\sum_{k=1}^\infty\frac{\beta_{2k+1}}{(\xi+\lambda_{2k+1})^2+\tau^2}-$$

$$-\tau\sum_{k=1}^\infty\frac{\beta_{2k}}{(\xi+\lambda_{2k})^2+\tau^2}.$$

We know that

$$(\xi+\lambda_k)^2+\tau^2\le[(\xi+\lambda_k)^2+1](1+\tau^2).$$

Then we get

$$\frac{1}{(\xi+\lambda_k)^2+\tau^2}\ge\frac{1}{1+\tau^2}\,\frac{1}{(\xi+\lambda_k)^2+1},\tag{14}$$

and

$$\frac{1}{\xi^2+\tau^2}\ge\frac{1}{1+\tau^2}\,\frac{1}{1+\xi^2}.\tag{15}$$

Thus, according to (14) and (15), we can obtain the following estimates

$$|\mathrm{Re}\widetilde{K}(\xi+i\,\tau)| = \frac{1}{\sqrt{\pi}}\frac{(1-\varepsilon)\,\xi}{\xi^2+\tau^2} + \sum_{k=0}^{\infty}\frac{\beta_{2k+1}\,(\xi+\lambda_{2k+1})}{(\xi+\lambda_{2k+1})^2+\tau^2} +$$

$$+\sum_{k=1}^{\infty}\frac{\beta_{2k}\,(\xi+\lambda_{2k})}{(\xi+\lambda_{2k})^2+\tau^2} \geq$$

$$\geq \frac{1}{1+\tau^2}\left(\frac{1}{\sqrt{\pi}}\frac{(1-\varepsilon)\,\xi}{1+\xi^2} + \sum_{k=0}^{\infty}\frac{\beta_{2k+1}\,(\xi+\lambda_{2k+1})}{(\xi+\lambda_{2k+1})^2+1}\right) = \frac{C_{1,\xi}}{1+\tau^2}, \tag{16}$$

and

$$|\mathrm{Im}\widetilde{K}(\xi+i\,\tau)| = |\tau|\,\frac{1}{\sqrt{\pi}}\frac{1-\varepsilon}{\xi^2+\tau^2} + |\tau|\,\sum_{k=0}^{\infty}\frac{\beta_{2k+1}}{(\xi+\lambda_{2k+1})^2+\tau^2} +$$

$$+|\tau|\,\sum_{k=1}^{\infty}\frac{\beta_{2k}}{(\xi+\lambda_{2k})^2+\tau^2} \geq$$

$$\geq \frac{|\tau|}{1+\tau^2}\left(\frac{1}{\sqrt{\pi}}\frac{1-\varepsilon}{1+\xi^2} + \sum_{k=0}^{\infty}\frac{\beta_{2k+1}}{(\xi+\lambda_{2k+1})^2+1}\right) = \frac{C_{2,\xi}\,|\tau|}{1+\tau^2}, \tag{17}$$

where $C_{1,\xi}$, $C_{2,\xi}$ are constants only depending on ξ and they are as follows:

$$C_{1,\xi} = \frac{1}{\sqrt{\pi}}\frac{(1-\varepsilon)\,\xi}{1+\xi^2} + \sum_{k=0}^{\infty}\frac{\beta_{2k+1}\,(\xi+\lambda_{2k+1})}{(\xi+\lambda_{2k+1})^2+1},$$

$$C_{2,\xi} = \frac{1}{\sqrt{\pi}}\frac{1-\varepsilon}{1+\xi^2} + \sum_{k=0}^{\infty}\frac{\beta_{2k+1}}{(\xi+\lambda_{2k+1})^2+1}.$$

From (16) and (17), we have the following estimate

$$|\widetilde{K}(\xi+i\,\tau)|^2 = |\mathrm{Re}\widetilde{K}(\xi+i\,\tau)|^2 + |\mathrm{Im}\widetilde{K}(\xi+i\,\tau)|^2 \geq$$

$$\geq \frac{\min(C_{1,\xi}^2, C_{2,\xi}^2)}{1+\tau^2},$$

or

$$|\widetilde{K}(\xi+i\,\tau)| \geq \frac{C_\xi}{\sqrt{1+\tau^2}}, \quad \text{where} \quad C_\xi = \min(C_{1,\xi}, C_{2,\xi}). \tag{18}$$

□

Then, proceed to the limit as $\xi \to 0$ from (13), we obtain the equality

$$\nu(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)}\,e^{i\,\tau t}\,d\tau. \tag{19}$$

Proposition 3.3. *([32]) Assume that $\phi \in W(M)$. Then for the imaginary part of the Laplace transform of function $\phi(t)$ the following inequality holds:*

$$\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)|\sqrt{1+\tau^2}\, d\tau \le C\, \|\phi\|_{W_2^2(R_+)},$$

where $C > 0$ is a constant.

Now we prove Theorem 3.1.

Proof of Theorem 3.1. First of all, we prove that $\nu \in W_2^1(\mathbb{R}_+)$. Due to (18) and (19), we get the estimate

$$\int_{-\infty}^{+\infty} |\widetilde{\nu}(\tau)|^2 (1+|\tau|^2)\, d\tau \;=\; \int_{-\infty}^{+\infty} \left| \frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)} \right|^2 (1+|\tau|^2)\, d\tau \;\le$$

$$\le C_0 \int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)|^2 (1+|\tau|^2)^2\, d\tau \;=\; C_0 \|\phi\|^2_{W_2^2(\mathbb{R})},$$

where $C_0 = \min(C_{1,0}, C_{2,0})$ is determined by (18).

Besides, we have

$$|\nu(t) - \nu(s)| \;=\; \left| \int_s^t \nu'(y)\, dy \right| \;\le\; \|\nu'\|_{L_2} (t-s)^{1/2}.$$

From (18), (19) and Proposition 3.3, we can write

$$|\nu(t)| \le \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\,\tau)|}{|\widetilde{K}(i\,\tau)|} d\tau \le$$

$$\le \frac{1}{2\pi C_0} \int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)|\sqrt{1+\tau^2} d\tau \le$$

$$\le \frac{C}{2\pi C_0} \|\phi\|_{W_2^2(R_+)} \le \frac{C\, M}{2\pi C_0} = 1,$$

where M is as follows:

$$M = \frac{2\pi C_0}{C}.$$

Conclusions

In this paper, the boundary control problem for a parabolic equation with involution was considered. The studied boundary control problem was reduced to the Volterra integral equation of the first kind by the Fourier method, and the existence of a solution of the integral equation was proved using the Laplace transform method. In the future research, we will continue to study boundary-value control problems for parabolic equations involving involution, including proving the existence of control function in two and $n-$dimensional domains.

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