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INVERSE PROBLEM FOR WAVE EQUATION OF MEMORY TYPE WITH ACOUSTIC BOUNDARY CONDITIONS

Totieva Z. D., Durdiev D. K.

Abstract In this article, for the direct problem, which consists of finding the velocity potential and the displacement of boundary points in the wave equation of memory type with initial and boundary acoustic conditions, the inverse problem of determining the memory kernel by the integral overdetermination condition is studied. By introducing a new function, the problem is reduced to a problem with homogeneous boundary conditions. Using the technique of estimating integral equations and the contraction mappings principle in Sobolev spaces, the existence and uniqueness theorem for the inverse problem is proved.

Key words: wave equation, acoustic boundary condition, inverse problem, memory kernel, contraction mapping principle.

AMS Mathematics Subject Classification: 35L20, 35R30, 35Q99.

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1 Introduction

We consider a wave equation of memory type with initial and acoustic boundary conditions:

$$u_{tt} - u_{xx} + \int_0^t k(t-s)u_{xx}(x,s)\,ds = 0,\ \ (x,t) \in I \times (0,T), \tag{1.1}$$

$$u\Big|_{x=0} \equiv 0, \quad 0 < t < T, \tag{1.2}$$

$$\left[u_x - \int_0^t k(s)u_x(x,t-s)\,ds\right]\Bigg|_{x=l} = y'(t), \quad 0 < t < T, \tag{1.3}$$

$$u_t\Big|_{x=l} = -py'(t) - qy(t), \quad 0 < t < T, \tag{1.4}$$

$$u\Big|_{t=0} = u_0(x),\ u_t\Big|_{t=0} = u_1(x), \quad x \in I, \tag{1.5}$$

where $u(x,t)$ is the velocity potential, $k(t)$ is the memory kernel, $y(t)$ is displacement at the point $x = l$; $T,\ p,\ q$ are some positive known constants, $I := (0,l)$.

Recently, wave equations with acoustic boundary conditions have been studied by many authors. In [1], the problem (1.1)-(1.5) was considered in the multidimensional case, when I is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, Γ_0 and Γ_1 are closed and disjoint. In equation (1.3), the derivative with respect to the unit outward normal to Γ is used. This work was devoted to investigating the influence of the kernel function k and proving general rates of decay of solutions when k does not decay exponentially. In [2] authors consider the nonlinear viscoelastic Kirchhoff-type equation with initial conditions and acoustic boundary conditions. They show that, depending on the properties of convolution kernel k at infinity, the energy of the solution decays exponentially or polynomially as $t \to +\infty$. The approach is based on integral inequalities and multiplier techniques. Instead of using a Lyapunov-type technique for some perturbed energy, authors concentrate on the original energy, showing that it satisfies a nonlinear integral inequality which, in turn, yields the final decay estimate.

Boundary conditions of memory type, as (1.3), imposed on a portion of the boundary and Dirichlet condition on the other part of the boundary, have been considered, for example in [3]-[5]. Condition (1.3) means that the right end of I is clamped in a body with viscoelastic properties. For detailed analyzes in this direction, the reader is referred to articles in references of [1], [2].

The kernel of memory term in (1.1) represents the physical properties of viscoelasticity medium, which is difficult to determine directly. By taking additional information on the velocity potential u, we can reconstruct k, provided the velocity potential is taken on a suitable subset of the I. We suppose that such an additional information on u can be represented in integral form, called integral overdetermination condition, by

$$\int_0^l \varphi(x)u_x(x,t)dx = f(t), \quad t \in [0,T], \tag{1.6}$$

where $f(t)$ is the measurement data, which represents the average velocity on I and φ is a given function representing the type of device used to measure the velocity $u_x(x,t)$ and $supp(\varphi) \subseteq I$.

If function $k(t)$ is known, then the initial-boundary value problem (1.1)–(1.5) of determining $u(x,t)$ and $y(t)$ is called *the direct problem*. *The inverse problem* is the problem of determining $u(x,t)$, $y(t)$ and $k(t)$ from a system of partial integro-differential equations (1.1)–(1.6).

This study relates to the class of one dimensional inverse problems in the linear dynamical wave processes of memory type. The unknown function in the above problem is the kernel of the integral operator modeling the memory phenomenon, which is relevant when wave processes propagate in media.

In the theory of inverse problems for a hyperbolic integro-differential equations, those of finding the integral-operator kernel (dependent on time and spatial variables) comprise a field that shaped at the end of the last century [6]-[11]. A more detailed analysis of the literature on this subject is available in [12], which is one of the most recent fundamental papers in the field of inverse problems for media with after-effect. It contains the results of studying the well-posedness of a number of one- and multi-dimensional inverse dynamical problems for hyperbolic integro-differential equations that occur when internal characteristics of media with aftereffects are described using measurements of the wave field in accessible domains.

Multidimensional inverse problems are poorly studied. Its study is of both mathematical interest and interest in applications. The problems of determining kernel which depends on two or more variables are considered in [13]-[22]. In [13], the method of separation of variables is used to solve inverse problems in a bounded domain, by which the problems are reduced to a system of integral equations of the Volterra type with respect to unknown functions depending on a time variable. In [14]-[19], multidimensional inverse problems with concentrated sources of perturbation are studied. These problems are reduced to solving problems of integral geometry. The purposes of inverse problems is to determine the spatial parts of kernel and Lame coefficients.

In [20], a local unique solvability of the problem of determining the kernel of $k(x,t)$ was obtained based on the method of scales of Banach spaces for the viscoelasticity equation in the class of functions analytic in the variable x and smooth in the variable t. Later in [21], [22], using the same method in combination with the method of weight norms, the global unique solvability of the problem of determining $k(x,t)$ was studied.

In recent years, we witness an increase in the number of publications on numerical calculations of integral operator kernels [23]-[26].

The theoretical significance of the study is to obtain the necessary and sufficient conditions for the local unique solvability of the onedimensional inverse problem (1.1)-(1.6).

Next, it is more convenient to study the inverse problem (1.1)-(1.6) in terms of a functions $v := u_t + z\frac{x}{l},\ z := py' + qy$. Equations (1.1)-(1.6) in terms of functions v, z are below in section 2.2. The function v has zero boundary conditions, which is necessary when applying the energy inequality in section 4.

2 The concrete version of the inverse problem

2.1 Function spaces. For any integers m, p we denote by $W^{m,p}(I) := W^{m,p}(I;\mathbb{R})$ and $W^{m,p}(0,T) := W^{m,p}(0,T;\mathbb{R})$ for the usual Sobolev spaces defined for spatial variable and time variable respectively. For the Banach space $\mathbb{X}$, the space $L^p(0,T;\mathbb{X})$ consist all Lebesgue measurable functions $u : [0,T] \to \mathbb{X}$ with

$$\|u\|_{L^p(0,T;\mathbb{X})} := \left(\int_0^T \|u(t)\|_{\mathbb{X}}^p \, dt\right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

The Sobolev spaces $W^{m,p}(0,T;\mathbb{X})$ consists all functions $u \in L^p(0,T;\mathbb{X})$ such that $\frac{\partial^\gamma u}{\partial t}$ exists in the weak sense and belongs to $L^p(0,T;\mathbb{X})$ for all $0 \le \gamma \le m$. Let $H^m(I) := W^{m,2}(I),\ H^m(0,T) := W^{m,2}(0,T)$, and $H^m(0,T;\mathbb{X}) := W^{m,2}(0,T;\mathbb{X})$.

Let us represent

$$H_0^1(I) = \{\varphi \in H^1(I) : \varphi(0) = \varphi(l) = 0\}.$$

$$H_0^3(I) = \{\varphi \in H^3(I) : \varphi(0) = \varphi(l) = 0, \varphi'(0) = \varphi'(l) = 0, \varphi''(0) = \varphi''(l) = 0\}.$$

2.2 The inverse problem. For $T > 0$, the inverse problem is to determine $\tau \in (0,T]$,

$$u \in H^3(0,\tau; H^2(I)),\ k \in H^1(0,\tau),\ \text{and } y \in H^3(0,\tau) \tag{2.1}$$

or

$$v \in H^2(0,\tau; H_0^1(I) \cap H^2(I)),\ k \in H^1(0,\tau),\ \text{and } y \in H^3(0,\tau) \tag{2.2}$$

such that (v,k,z) satisfies the system:

$$v_{tt} - v_{xx} + k(t)u_0''(x) + \int_0^t k(t-s)v_{xx}(x,s)\,ds = z''\frac{x}{l},\ (x,t) \in I \times (0,\tau), \tag{2.3}$$

$$v\big|_{x=0} = v\big|_{x=l} = 0, \quad 0 \le t < \tau, \tag{2.4}$$

$$\left[v_x - \int_0^t k(t-s)v_x(x,s)\,ds\right]_{x=l}$$

$$= y''(t) + k(t)u_0'(l) + \frac{1}{l}\left(z(t) - \int_0^t k(t-s)z(s)\,ds\right), \quad 0 < t < \tau, \tag{2.5}$$

$$v\big|_{t=0} = v_0(x),\ v_t\big|_{t=0} = v_1(x), \quad x \in I, \tag{2.6}$$

$$\int_0^l \varphi'(x)\left[v_x - \frac{z}{l}\right]dx = f'(t), \quad t \in (0,\tau), \tag{2.7}$$

under following assumption on the data:
$(i1)$ $u_0(x), u_1(x) \in H^4(I), \varphi(x) \in H_0^3(I)$.
$(i2)$ $v_0(x) := u_1(x) - u_1(l)\frac{x}{l}$ and $v_1(x) := u_0''(x) - u_0''(l)\frac{x}{l} \in H_0^1(I) \cap H^2(I)$.
$(i3)$ $\alpha^{-1} := \int_0^l \varphi'(x)u_0''(x)dx \neq 0$.
$(i4)$ $f(t) \in H^4(0,T)$:
$\int_0^l \varphi(x)u_0'(x)dx = -\int_0^l \varphi'(x)u_0(x)dx = f(0)$,
$\int_0^l \varphi(x)u_1'(x)dx = -\int_0^l \varphi'(x)u_1(x)dx = f'(0)$,
$\int_0^l \varphi(x)u_0''(x)dx = -\int_0^l \varphi'(x)u_0'(x)dx = f''(0)$.
$-\int_0^l \varphi'(x)[u_1''(x) - k(0)u_0''(x)]dx = f'''(0)$ with $k(0) = \alpha(f'''(0) - \int_0^l v_0(x)\varphi'''(x)dx)$.
$(i5)$ $y''(0) = \frac{1}{\psi(l)}\left[(1-k(0))(f'(0)) + \int_0^l \psi(x)(u_1''(x) - k(0)u_0''(x))dx\right]$
with $\psi(x) = \int_0^x \varphi(x)dx$ and $\psi(l) \neq 0$.

Remark 1. By the Sobolev's embedding theorem, if $j + 1/2 < s,\ h(t) \in H^s(0,T)$, then $h(t) \in C^j[0,T]$ and from this the equalities in $(i4)$ and $(i5)$ are correct.

Remark 2. Overdetermination condition (1.6) may be rewritten as:

$$-\int_0^l \varphi'(x)u(x,t)dt = f(t), \quad t \in (0,\tau) \tag{2.7$'$}$$

or

$$\psi(l)u_x(l,t) - \int_0^l \psi(x)u_{xx}dx = f(t), \tag{2.7$''$}$$

where $\psi(x)$ is determined in $(i5)$.

Then we find for the equality (2.5) using (2.7”):

$$y''(t) = G[v_{xx}](t) - k(t)G[u_{xx}](0) - \int_0^t k(t-s)G[v_{xx}](s)ds, \tag{2.8}$$

$G[\cdot](t) := \frac{1}{\psi(l)}\left(f'(t) + \int_0^l \psi(x)[\cdot](x,t)dx\right)$.

2.3. Important inequalities and results. The following inequalities are used in this paper.

Lemma 1. (Theorem 4.4, [27]). *Let X be a Banach space, $p \in (1,\infty)$, $\tau \in \mathbb{R}_+$, $k \in L^p(0,\tau)$, and $f \in L^p(0,\tau;X)$. Then $k * f \in L^p(0,\tau;X)$ and*

$$\|k * f\|_{L^p(0,\tau;X)} \leq \tau^{1-1/p}\|k\|_{L^p(0,\tau)}\|f\|_{L^p(0,\tau;X)},$$

where $(k*f)(t) = \int_0^t k(t-s)f(s)ds$.

Proof. Using Young's inequality for convolution and Holder's inequality, we have

$$\|k * f\|_{L_p(0,\tau;X)} \leq \|k\|_{L^1(0,\tau)}\|f\|_{L^p(0,\tau;X)} \leq \tau^{1-1/p}\|k\|_{L^p(0,\tau)}\|f\|_{L^p(0,\tau;X},$$

which completes the proof.

Lemma 2. (Theorem 4.5, [27]).*Let X be a Banach space, $p \in (1,\infty), \tau \in \mathbb{R}_+$, $w \in W^{1,p}(0,\tau;X)$ with $w(0) = 0$. Then*

$$\|w\|_{L^\infty(0,\tau;X)} \leq \tau^{1-1/p}\|w_t\|_{L^p(0,\tau;X)},$$

$$\|w\|_{L^p(0,\tau;X)} \leq \tau\|w_t\|_{L^p(0,\tau;X)}.$$

The proof can be easily concluded from Holder's inequality and Young's inequality for convolution, using the formula $w = 1 * w_t$ with $w(0) = 0$.

3 The Equivalent Problem

Lemma 3. Let the assumption $(i1)-(i5)$ hold and (v,k,y) with $v = u_t + z\frac{x}{l}$ be a solution of the system (2.2)-(2.7) defined up to T such that (2.1) is hold.

Then (v,k,y) satisfy the conditions:

$$v \in H^2(0,\tau;H_0^1(I)\cap H^2(I)),\ k \in H^1(0,T), y \in H^3(0,T) \tag{3.1}$$

and solve the system

$$v_{tt} - v_{xx} + k(t)u_0''(x) + \int_0^t k(t-s)v_{xx}(x,s)\,ds = z''\frac{x}{l},\ (x,t) \in I\times(0,T), \tag{3.2}$$

$$v\big|_{x=0} = v\big|_{x=l} = 0, \quad 0<t<T, \tag{3.3}$$

$$v\big|_{t=0} = v_0(x),\ v_t\big|_{t=0} = v_1(x), \quad x \in I, \tag{3.4}$$

with

$$\begin{aligned} k'(t) = \alpha\Bigg\{ f^{(IV)}(t) - \int_0^l v_t(x,t)\varphi'''(x)dx - k(0)\int_0^l v(x,t)\varphi'''(x)dx \\ - \int_0^t\int_0^l k'(t-s)v(x,s)\varphi'''(x)dxds\Bigg\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} y'''(t) = G'[v_{xx}](t) - k'(t)G[u_{xx}](0) - k(0)G[v_{xx}](t) \\ - \int_0^t k'(t-s)G[v_{xx}](s)\,ds, \quad 0<t<T, \end{aligned} \tag{3.6}$$

where $z'' = py''' + qy''$.

Proof.

Step 1. Suppose that the system (1.1)-(1.6) has a solution (u, k, y) satisfying (2.1). Then, it is easy to see that (v, k, y) satisfy the condition (3.1) and equations (3.2), (3.3).

From the equations (1.1), (1.5), we obtain

$$v(x,0) = u_1(x) - u_1(l)\frac{x}{l} = v_0(x),$$
$$u_{tt}(x,0) = u_{xx}(x,0) = u_0''(x),$$
$$v_t(x,0) = u_0''(x) - u_0''(l)\frac{x}{l} = v_1(x).$$

Therefore, we have (3.4).

Taking the inner product in (3.2) with φ' and using assumption $(i1)$ and $(i3)$

$$k(t) = \alpha\left\{f'''(t) - \int_0^l v(x,t)\varphi'''(x)dx - \int_0^t\int_0^l k(t-s)v(x,s)\varphi'''(x)dxds\right\}, \qquad (3.7)$$

where $f'''(t) = -\int_0^l \varphi'(x)\left(v_{tt} - z''\frac{x}{l}\right)dx$ and after differentiating by t the equation (3.5) follows.

Using (2.8) in Remark 2 and then after differentiating it by t the equation (3.6) follows.

Step 2. Suppose that the system (3.2)–(3.6) has a solution (v, k', z'') satisfying (3.1). It can be easily shown that (2.1) and (1.2)–(1.5) hold. Since $v = u_t + z\frac{x}{l}$, the equation (3.2) can be rewritten as

$$\frac{\partial}{\partial t}\left(u_{tt} + z'\frac{x}{l} - u_{xx} - \int_0^t k(s)u_{xx}(x,t-s)ds\right) = 0.$$

Integrating the above equation we obtain

$$u_{tt} + z'\frac{x}{l} - u_{xx} - \int_0^t k(t-s)u_{xx}ds = const_1.$$

Taking $t = 0$ we have $const_1 = 0$. Hence, we obtain (1.1).

The equation (3.5) for k' can be rewritten as

$$f^{(IV)}(t) = \frac{\partial}{\partial t}\left(-\int_0^l \varphi'(x)u_{xxt}(x,t)dx + k(0)\int_0^l u_{xx}\varphi'(x)dx\right.$$
$$\left.+\int_0^t\int_0^l k'(t-s)\varphi'(x)u_{xx}(x,s)dxds\right).$$

Integrating this equation with respect to t, we find

$$f'''(t) = -\int_0^l \varphi'(x)u_{xxt}(x,t)dx + k(0)\int_0^l u_{xx}\varphi'(x)dx$$
$$+\int_0^t\int_a^b k'(t-s)\varphi'(x)u_{xx}(x,s)dxds + const_2,$$

Taking $t = 0$ and using assumption $(i4)$, we get $const_2 = 0$, hence

$$f'''(t) = -\int_0^l \varphi'(x)u_{xxt}(x,t)dx + k(0)\int_0^l u_{xx}(t,x)\varphi'(x)dx$$

$$+\int_0^t\int_a^b k'(t-s)\varphi'(x)u_{xx}(x,s)dxds,$$

at the same time

$$f'''(t) = \frac{\partial}{\partial t}\left(-\int_0^l \varphi'(x)u_{xx}(x,t)dx + \int_0^t\int_a^b k(t-s)\varphi'(x)u_{xx}(x,s)dxds\right).$$

Integrating with respect to t, we have

$$f''(t) = -\int_0^l \varphi'(x)u_{xx}(x,t)dx + \int_0^t\int_0^l k(t-s)\varphi'(x)u_{xx}(x,s)dxds + const_3.$$

Taking $t = 0$ and using assumption $(i4)$, we get $const_3 = 0$, hence

$$f''(t) = -\int_0^l \varphi'(x)u_{xx}(x,t)dx + \int_0^t\int_0^l k(t-s)\varphi'(x)u_{xx}(x,s)dxds.$$

Using (1.1) in the above equation along with assumption $(i1)$, we have

$$f''(t) = -\int_0^l \varphi'(x)u_{tt}dx = -\frac{\partial}{\partial t}\int_0^l \varphi'(x)u_t dx.$$

Integrating twice the above equation, taking $t = 0$ and using assumption $(i4)$ we find

$$f(t) = -\int_0^l \varphi'(x)u(x,t)dx,$$

which is $(2.7')$ and hence (1.6).

The equivalence of equalities (3.6) and (2.5) is proved in a similar way using assumption $(i5)$. Lemma 3 is proved.

4 The Main Result

From equivalent problem (3.1)-(3.6) we see that unknown function $y'''(t)$ can easily be excluded from the system (3.1)-(3.6), since it is expressed in terms of k and v. Therefore, the main result will be proved for functions k, v.

Theorem 1. *Let assumption $(i1) - (i5)$ hold. Then there exist $\tau \in (0,T)$, such that the inverse problem (2.1)-(2.8) has a unique solution*

$$(v,k) \in H^3(0,\tau; H_0^1(I)\cap H^2(I)) \times H^1(0,\tau).$$

Proof. Let $Q(\tau, M)$ be the space of functions $(\tilde{v}, \tilde{k}') \in H^2(0,\tau; H_0^1(I) \cap H^2(I)) \times L^2(0,\tau)$ such that

$$\|\tilde{v}\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))} + \|\tilde{k}'\|_{L^2(0,\tau)} \leq M$$

and (3.4)-(3.6) hold. $M = const > 0$ will be determined later.

We define the mapping $A : Q(\tau, M) \to Q(\tau, M)$ such that $(\tilde{v}, \tilde{k}') \to (v, k')$ through

$$k'(t) = \alpha\left\{ f^{(IV)}(t) - \int_0^l \tilde{v}_t(x,t)\varphi'''(x)dx - k(0)\int_0^l \tilde{v}(x,t)\varphi'''(x)dx \right.$$
$$\left. - \int_0^t \int_0^l \tilde{k}'(t-s)\tilde{v}(x,s)\varphi'''(x)dxds \right\}, \tag{4.1}$$

and the initial boundary value problem

$$v_{tt} - v_{xx} + \tilde{k}(t)u_0''(x) + \int_0^t \tilde{k}(t-s)\tilde{v}_{xx}(x,s)\,ds = z''\frac{x}{l},\ (x,t) \in I \times (0,T),$$
$$v\big|_{x=0} = v\big|_{x=l} = 0, \quad 0 < t < T, \tag{4.2}$$
$$v\big|_{t=0} = v_0(x),\ v_t\big|_{t=0} = v_1(x), \quad x \in I$$

with

$$y'''(t) = G'[\tilde{v}_{xx}](t) - k'(t)G[u_{xx}](0) - k(0)G[\tilde{v}_{xx}](t)$$
$$- \int_0^t \tilde{k}'(t-s)G[\tilde{v}_{xx}](s)\,ds, \quad 0 < t < T. \tag{4.3}$$

In above equalities $\tilde{k}(t) = k(0) + \int_0^t \tilde{k}'(s)ds$. Show that the mapping $A : Q(\tau, M) \to Q(\tau, M)$ is a contraction map.

Step 1. Firstly, we show that the map A is well defined for an appropriate choice of M and τ. So, from (4.1) we find

$$\|k'\|_{L^2(0,\tau)} \leq \alpha\left\{ \|f^{(IV)}(t)\|_{L^2(0,\tau)} + \|\tilde{v}_t\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)} \right.$$
$$\left. +|k(0)|\|\tilde{v}\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)} + \left\|\int_0^t \int_0^l \tilde{k}'(t-s)\tilde{v}(x,s)\varphi'''(x)dxds\right\|_{L^2(0,\tau)}\right. . \tag{4.4}$$

Using the inequality (Lemma 1)

$$\|k * f\|_{L^2(0,\tau;L^2(I))} \leq \tau^{1/2}\|k\|_{L^2(0,\tau)}\|f\|_{L^2(0,\tau;L^2(I))},$$

we have

$$\left\|\int_0^t \int_a^b \tilde{k}'(t-s)\tilde{v}(x,s)\varphi'''(x)dxds\right\|_{L^2(0,\tau)} \leq \tau^{1/2}\|\tilde{k}'\|_{L^2(0,\tau)}\left\|\int_a^b \tilde{v}(x,t)\varphi'''(x)dx\right\|_{L^2(0,\tau)}$$
$$\leq \tau^{1/2}\|\tilde{k}'\|_{L^2(0,\tau)}\left(\int_0^\tau \|\tilde{v}(t)\|^2_{L^2(I)}\|\varphi'''\|^2_{L^2(I)}dt\right)^{1/2}$$
$$= \tau^{1/2}\|\tilde{k}'\|_{L^2(0,\tau)}\|\varphi'''\|_{L^2(I)}\|\tilde{v}\|_{L^2(0,\tau;L^2(I))}. \tag{4.7}$$

Applying Lemma 2 implies

$$\begin{aligned}
\|\tilde{v}\|_{L^2(0,\tau;L^2(I))} &\le \|\tilde{v}-v_0\|_{L^2(0,\tau;L^2(I))} + \|v_0\|_{L^2(0,\tau;L^2(I))} \\
&\le \tau\|(\tilde{v}-v_0)_t\|_{L^2(0,\tau;L^2(I))} + \tau^{1/2}\|v_0\|_{L^2(I)} \\
= \tau\|\tilde{v}_t\|_{L^2(0,\tau;L^2(I))} &+ \tau^{1/2}\|v_0\|_{L^2(I)} \le \tau(\tau\|\tilde{v}_{tt}\|_{L^2(0,\tau;L^2(I))} + \tau^{1/2}\|v_1\|_{L^2(I)}) \\
+\tau^{1/2}\|v_0\|_{L^2(I)} &= \tau^2\|\tilde{v}_{tt}\|_{L^2(0,\tau;L^2(I))} + \tau^{3/2}\|v_1\|_{L^2(I)} + \tau^{1/2}\|v_0\|_{L^2(I)},
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\|\tilde{v}_t\|_{L^2(0,\tau;L^2(I))} &\le \|\tilde{v}_t-v_1\|_{L^2(0,\tau;L^2(I))} + \|v_1\|_{L^2(0,\tau;L^2(I))} \\
&\le \tau\|(\tilde{v}_t-v_1)_t\|_{L^2(0,\tau;L^2(I))} + \tau^{1/2}\|v_1\|_{L^2(I)} \\
&= \tau\|\tilde{v}_{tt}\|_{L^2(0,\tau;L^2(I))} + \tau^{1/2}\|v_1\|_{L^2(I)},
\end{aligned} \tag{4.9}$$

Using (4.5)-(4.9) in (4.4) and then by the definition of the space $Q(\tau, M)$, we arrive at:

$$\begin{aligned}
\|k'\|_{L^2(0,\tau)} &\le \alpha\Big\{\|f^{(IV)}(t)\|_{L^2(0,\tau)} + \|\varphi'''\|_{L^2(I)}\Big[\tau\|\tilde{v}_{tt}\|_{L^2(0,\tau;L^2(I))} + \tau^{1/2}\|v_1\|_{L^2(I)} \\
&+(\tau^2\|\tilde{v}_{tt}\|_{L^2(0,\tau;L^2(I))} + \tau^{3/2}\|v_1\|_{L^2(I)} + \tau^{1/2}\|v_0\|_{L^2(I)})(|k(0)| + \tau^{1/2}\|\tilde{k}'\|_{L^2(0,\tau)})\Big]\Big\} \\
&\le \alpha\Big\{\|f^{(IV)}(t)\|_{L^2(0,\tau)} + \|\varphi'''\|_{L^2(I)}\Big[\tau M + \tau^{1/2}\|v_1\|_{L^2(I)} \\
&+(\tau^2 M + \tau^{3/2}\|v_1\|_{L^2(I)} + \tau^{1/2}\|v_0\|_{L^2(I)})(|k(0)| + \tau^{1/2}M)\Big]\Big\}.
\end{aligned} \tag{4.10}$$

Next we estimate y''' from (4.3):

$$\begin{aligned}
\|y'''\|_{L^2(0,\tau)} &\le \|G'[\tilde{v}_{xx}](t)\|_{L^2(0,\tau)} + |G[u_{xx}](0)|\|k'\|_{L^2(0,\tau)} + |k(0)|\|G[\tilde{v}_{xx}]\|_{L^2(0,\tau)} \\
&+\tau^{1/2}\|\tilde{k}'\|_{L^2(0,\tau)}\|G[v_{xx}]\|_{L^2(0,\tau)} \le \frac{1}{|\psi(l)|}\Big(\|f''\|_{L^2(0,\tau)}| \\
&+\|\psi\|_{L^2(I)}(\tau\|\tilde{v}\|_{H^2(0,\tau;H^2(I))} + \tau^{1/2}\|v_1\|_{L^2(I)})\Big) + |G[u_{xx}](0)|\|k'\|_{L^2(0,\tau)} \\
&+\frac{1}{|\psi(l)|}(|k(0)| + \tau^{1/2}\|\tilde{k}'\|_{L^2(0,\tau)})\Big(\|f'\|_{L^2(0,\tau)} \\
&+\|\psi\|_{L^2(I)}(\tau^2\|\tilde{v}\|_{H^2(0,\tau;H^2(I))} + \tau^{3/2}\|v_1\|_{L^2(I)} + \tau^{1/2}\|v_0\|_{L^2(I)})\Big) \\
&\le \frac{1}{|\psi(l)|}\Big(\|f''\|_{L^2(0,\tau)}| + \|\psi\|_{L^2(I)}(\tau M + \tau^{1/2}\|v_1\|_{L^2(I)})\Big) \\
&+|G[u_{xx}](0)|\|k'\|_{L^2(0,\tau)} + \frac{1}{|\psi(l)|}(|k(0)| + \tau^{1/2}M)\Big(\|f'\|_{L^2(0,\tau)} \\
&+\|\psi\|_{L^2(I)}(\tau^2 M + \tau^{3/2}\|v_1\|_{L^2(I)} + \tau^{1/2}\|v_0\|_{L^2(I)})\Big).
\end{aligned} \tag{4.11}$$

Then we will use (4.10) for $\|k'\|_{L^2(0,\tau)}$ in (4.11).

For simplicity, hereafter C represents a generic constant depending only on I.

Using the energy estimate for the linear problem (4.2), we have

$$\|v\|_{H^2(0,\tau;H^2(I))} \le C(\|v_0\|_{H_0^1(I)\cap H^2(I)} + \|v_1\|_{H_0^1(I)} + \|G\|_{L^2(0,\tau;L^2(I))}),$$

where

$$G := -\tilde{k}(t)u_0''(x) - \int_0^t \tilde{k}(t-s)\tilde{v}_{xx}(x,s)\,ds + z''\frac{x}{l}.$$

Next, we estimate G as follows

$$\|G\|_{L^2(0,\tau;L^2(I))} \le \|\tilde{k}\|_{L^2(0,\tau)}\|u_0''\|_{L^2(I)} + \tau^{1/2}\|\tilde{k}\|_{L^2(0,\tau)}\|\tilde{v}_{xx}\|_{L^2(0,\tau;L^2(I))} + \|z''\|_{L^2(0,\tau)}$$
$$\le C\Big((\|u_0\|_{H^2(I)} + \tau^{1/2}\|\tilde{v}\|_{L^2(0,\tau;H^2(I))})\|\tilde{k}\|_{L^2(0,\tau)} + \|z''\|_{L^2(0,\tau)}\Big).$$

Thus, making use of definition of the space $Q(\tau, M)$ and using $y''(t) = y''(0) + \int_0^t y'''(s)ds$ we obtain the following estimate

$$\|G\|_{L^2(0,\tau;L^2(I))} \le C\Bigg\{ \left(\|u_0\|_{H^2(I)} + \tau^{1/2}M\right) \left\|k(0) + \int_0^t \tilde{k}'(s)ds\right\|_{L^2(0,\tau)} + p\|y'''\|_{L^2(0,\tau)}$$
$$+q\left\|y''(0) + \int_0^t y'''(s)ds\right\|_{L^2(0,\tau)} \Bigg\} \le C\Bigg\{ \left(\|u_0\|_{H^2(I)} + \tau^{1/2}M\right)\left(\tau^{1/2}|k(0)| + \tau\|\tilde{k}'\|_{L^2(0,\tau)}\right)$$
$$+p\|y'''\|_{L^2(0,\tau)} + q\tau^{1/2}|y''(0)| + q\tau\|y'''\|_{L^2(0,\tau)}\Bigg\}.$$

Then using (4.10) and (4.11) we have

$$\|G\|_{L^2(0,\tau;L^2(I))} \le C\Bigg[\left(\|u_0\|_{H^2(I)} + \tau^{1/2}M\right)\left(\tau^{1/2}|k(0)| + \tau M\right)$$
$$+q\tau^{1/2}|y''(0)| + (p+\tau q)\|y'''\|_{L^2(0,\tau)}\Bigg], \tag{4.12}$$

If we take $\tau > 0$ small enough such that

$$\tau^2(1+M) \le 1, \tag{4.13}$$

estimates (4.10)-(4.12), transform to

$$
\begin{aligned}
&\|v\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))} + \|k'\|_{L^2(0,\tau)} \leq \\
&C\Big\{ \|v_0\|_{H_0^1(I)\cap H^2(I)} + \|v_1\|_{H_0^1(I)} + \big(\|u_0\|_{H^2(I)} + \tau^{1/2}M\big)\big(\tau^{1/2}|k(0)| + \tau M\big) \\
&\quad + q\tau^{1/2}|y''(0)| + (p+\tau q)\|y'''\|_{L^2(0,\tau)} + \|k'\|_{L^2(0,\tau)} \Big\} \\
&\leq C\Big\{ \|v_0\|_{H_0^1(I)\cap H^2(I)} + \|v_1\|_{H_0^1(I)} + \big(\|u_0\|_{H^2(I)} + 1\big)(|k(0)|+1) \\
&\quad + q|y''(0)| + (p+q)\bigg(\frac{1}{|\psi(l)|}\Big(\|f''\|_{L^2(0,\tau)}| + \|\psi\|_{L^2(I)}(1+\|v_1\|_{L^2(I)})\Big) \\
&\quad + \frac{1}{|\psi(l)|}(|k(0)|+1)\Big(\|f'\|_{L^2(0,\tau)} \\
&\quad + \|\psi\|_{L^2(I)}(1+\|v_1\|_{L^2(I)} + \|v_0\|_{L^2(I)})\Big) + |G[u_{xx}(0)]|F(\tau)\bigg) + F(\tau)\Big\} =: M_0.
\end{aligned}
$$

Here

$$
\begin{aligned}
F(\tau) = \alpha\bigg[&\|f^{(IV)}\|_{L^2(0,\tau)} \\
&+ \|\varphi'''\|_{L^2(I)}\Big[1 + \|v_1\|_{L^2(I)} + (1+\|v_1\|_{L^2(I)} + \|v_0\|_{L^2(I)})(|k(0)|+1)\Big]\bigg].
\end{aligned}
$$

We notice that $M_0 = M_0(\tau)$ is non-decreasing. Therefore, we can define a mapping $A : Q(\tau, M) \to Q(\tau, M)$ by choosing $M \geq M_0$ and τ small enough as in (4.13).

Step 2. In this step, we prove that A is a contraction mapping.

Let $(\tilde{v}_j, \tilde{k}'_j) \in Q(\tau, M)$ for $j = 1, 2$; define k'_j and v_j by (4.1)-(4.3) with $(\tilde{v}, \tilde{k}') = (\tilde{v}_j, \tilde{k}'_j)$ respectively. Then $(\tilde{v}_1, \tilde{k}'_j)$ for $j = 1, 2$ satisfies

$$
\begin{aligned}
&(v_1 - v_2)_{tt} - (v_1 - v_2)_{xx} + (\tilde{k}_1 - \tilde{k}_2)u_0''(x) \\
&+ \int_0^t (\tilde{k}_1 - \tilde{k}_2)(t-s)(\tilde{v}_2)_{xx}(x,s)\,ds + \int_0^t \tilde{k}_1(t-s)(\tilde{v}_1 - \tilde{v}_2)_{xx}(x,s)\,ds \\
&= (z_1'' - z_2'')\frac{x}{l}, \quad (x,t) \in I \times (0,T), \\
&v_1 - v_2\big|_{x=0} = v_1 - v_2\big|_{x=l} = 0, \quad 0 < t < T, \\
&v_1 - v_2\big|_{t=0} = 0, \ (v_1 - v_2)_t\big|_{t=0} = 0, \quad x \in I,
\end{aligned} \tag{4.14}
$$

with

$$(k_1'-k_2')(t)=-\alpha\bigg\{\int_0^l(\tilde v_1-\tilde v_2)_t\varphi'''(x)dx+k(0)\int_0^l(\tilde v_1-\tilde v_2)\varphi'''(x)dx$$
$$+\int_0^t\int_0^l(\tilde k_1-\tilde k_2)'(t-s)\tilde v_2(x,s)\varphi'''dxds+\int_0^t\int_0^l\tilde k_1'(t-s)(\tilde v_2-\tilde v_2)(x,s)\varphi'''dxds\bigg\}, \tag{4.15}$$

and

$$(y_1'''-y_2''')(t)=\frac{1}{\psi(l)}\int_0^l\psi(x)(\tilde v_1-\tilde v_2)_{txx}dx-(k_1'-k_2')(t)G[u_{xx}](0)$$
$$-\frac{k(0)}{\psi(l)}\int_0^l\psi(x)(\tilde v_1-\tilde v_2)_{xx}dx-\int_0^t(\tilde k_1'-\tilde k_2')(t-s)G[(\tilde v_2)_{xx}](s)\,ds \tag{4.16}$$
$$-\frac{1}{\psi(l)}\int_0^t\tilde k_1'(t-s)\int_0^l\psi(x)(\tilde v_1-\tilde v_2)_{xx}(x,s)\,dxds,\quad 0<t<T.$$

Now we estimate $k_1'-k_2'$ from the equation (4.15) as follows

$$\|k_1'-k_2'\|_{L^2(0,\tau)}\le\alpha\bigg\{\|(\tilde v_1-\tilde v_2)_t\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)}$$
$$+|k(0)|\|\tilde v_1-\tilde v_2\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)}+\left\|\int_0^t\int_0^l(\tilde k_1'-\tilde k_2')(t-s)\tilde v_2(x,s)\varphi'''(x)\,dxds\right\|_{L^2(0,\tau)}$$
$$+\left\|\int_0^t\int_0^l\tilde k_1'(t-s)(\tilde v_1-\tilde v_2)(x,s)\varphi'''(x)\,dxds\right\|_{L^2(0,\tau)}\bigg\}.$$

Next, we use the same argumentation as in (4.10):

$$\|k_1'-k_2'\|_{L^2(0,\tau)}\le\alpha\bigg\{\tau\|(\tilde v_1-\tilde v_2)_{tt}\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)}$$
$$+\tau^2|k(0)|\|(\tilde v_1-\tilde v_2)_{tt}\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)}$$
$$+\tau^{1/2}\|\tilde k_1'-\tilde k_2'\|_{L^2(0,\tau)}\|\varphi'''\|_{L^2(I)}\|\tilde v_2\|_{L^2(0,\tau;L^2(I))}$$
$$+\tau^{1/2}\|\tilde k_1'\|_{L^2(0,\tau)}\|\varphi'''\|_{L^2(I)}\|(\tilde v_1-\tilde v_2\|_{L^2(0,\tau;L^2(I))}\bigg\}$$
$$\le\alpha\tau^{1/2}\bigg\{\tau^{1/2}\|(\tilde v_1-\tilde v_2)_{tt}\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)}$$
$$+\tau^2|k(0)|\|(\tilde v_1-\tilde v_2)_{tt}\|_{L^2(0,\tau;L^2(I))}\|\varphi'''\|_{L^2(I)}+\tau^{1/2}\|\tilde k_1'-\tilde k_2'\|_{L^2(0,\tau)}\|\varphi'''\|_{L^2(I)}\|\tilde v_2\|_{L^2(0,\tau;L^2(I))}$$
$$+\tau^{1/2}\|\tilde k_1'\|_{L^2(0,\tau)}\|\varphi'''\|_{L^2(I)}\|(\tilde v_1-\tilde v_2\|_{L^2(0,\tau;L^2(I))}\bigg\}$$
$$\le\alpha\tau^{1/2}\|\varphi'''\|_{L^2(I)}\left(\tau^{1/2}+\tau^{3/2}|k(0)|+\tau^2M\right)\|(\tilde v_1-\tilde v_2\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))}$$
$$+\alpha\tau^{1/2}\|\varphi'''\|_{L^2(I)}M\|\tilde k_1'-\tilde k_2'\|_{L^2(0,\tau)}. \tag{4.17}$$

Further we have for (4.16)

$$\|y_1''' - y_2'''\|_{L^2(0,\tau)} \le \frac{1}{|\psi(l)|}\|\psi\|_{L^2(I)}\tau\|(\tilde v_1 - \tilde v_2)_{tt}\|_{L^2(0,\tau;L^2(I))} + |G[u_{xx}](0)|\|k_1' - k_2'\|_{L^2(0,\tau)}$$

$$+\frac{|k(0)|}{|\psi(l)|}\|\psi\|_{L^2(I)}\tau^2\|(\tilde v_1 - \tilde v_2)_{tt}\|_{L^2(0,\tau;L^2(I))} + \tau^{1/2}\|\tilde k_1' - \tilde k_2'\|_{L^2(0,\tau)}\|G[(\tilde v_2)_{xx}]\|_{L^2(0,\tau;L^2(I))}$$

$$+\frac{\tau^{1/2}}{|\psi(l)|}\|\psi\|_{L^2(I)}\|\tilde k_1'\|_{L^2(0,\tau)}\|(\tilde v_1 - \tilde v_2)_{xx}\|_{L^2(0,\tau;L^2(I))}\|\tilde k_1'\|_{L^2(0,\tau)}$$

$$\le \frac{\tau^{1/2}}{|\psi(l)|}\|\psi\|_{L^2(I)}\left(\tau^{1/2} + |k(0)|\tau^{3/2} + M\right)\|(\tilde v_1 - \tilde v_2\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))}$$

$$+\frac{\tau^{1/2}}{|\psi(l)|}\left(\|f'\|_{L^2(0,\tau)} + M\|\psi\|_{L^2(I)}\right)\|\tilde k_1' - \tilde k_2'\|_{L^2(0,\tau)}$$

$$+|G[u_{xx}](0)|\|k_1' - k_2'\|_{L^2(0,\tau)}. \tag{4.18}$$

Using the estimates for the linear problem (4.14), we estimate

$$\|v_1 - v_2\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))} \le C\|G_1\|_{L^2(0,\tau;L^2(I))},$$

where

$$G_1 := -(\tilde k_1 - \tilde k_2)u_0''(x) - \int_0^t (\tilde k_1 - \tilde k_2)(t-s)(\tilde v_2)_{xx}(x,s)\,ds$$

$$-\int_0^t \tilde k_1(t-s)(\tilde v_1 - \tilde v_2)_{xx}(x,s)\,ds + (z_1'' - z_2'')\frac{x}{l}.$$

For G_1 we obtain

$$\|G_1\|_{L^2(0,\tau;L^2(I))} \le \|\tilde k_1 - \tilde k_2\|_{L^2(0,\tau)}\|u_0\|_{H^2(I)} + \tau^{1/2}\|\tilde k_1 - \tilde k_2\|_{L^2(0,\tau)}\|(\tilde v_2)_{xx}\|_{L^2(0,\tau;L^2(I))}$$

$$+\tau^{1/2}\|\tilde k_1\|_{L^2(0,\tau)}\|(\tilde v_1 - \tilde v_2)_{xx}\|_{L^2(0,\tau;L^2(I))} + \|z_1'' - z_2''\|_{L^2(0,\tau)}\Big)$$

$$\le C\Big(\tau\|\tilde k_1' - \tilde k_2'\|_{L^2(0,\tau)}\|u_0\|_{H^2(I)} + \tau^{3/2}\|\tilde k_1' - \tilde k_2'\|_{L^2(0,\tau)}\|(\tilde v_2)\|_{L^2(0,\tau;H^2(I))}$$

$$+\tau^{3/2}\|\tilde k_1'\|_{L^2(0,\tau)}\|(\tilde v_1 - \tilde v_2)\|_{L^2(0,\tau;H_0^1(I)\cap H^2(I))} + \|z_1'' - z_2''\|_{L^2(0,\tau)}\Big)$$

$$\le C\Big(\tau\left(\|u_0\|_{H^2(I)} + \tau^{1/2}M\right)\|\tilde k_1' - \tilde k_2'\|_{L^2(0,\tau)}$$

$$+\tau^{3/2}M\|(\tilde v_1 - \tilde v_2)\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))} + (p + q\tau)\|y_1''' - y_2'''\|_{L^2(0,\tau)}\Big). \tag{4.19}$$

Thus, combining (4.17)-(4.19) we arrive at

$$\|v_1 - v_2\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))} + \|k_1' - k_2'\|_{L^2(0,\tau)} \le$$

$$\le C\Big(\tau\left(\|u_0\|_{H^2(I)} + \tau^{1/2}M\right)\|\tilde k_1' - \tilde k_2'\|_{L^2(0,\tau)}$$

$$+\tau^{3/2}M\|(\tilde v_1 - \tilde v_2)\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))} + (p + q\tau)\|y_1''' - y_2'''\|_{L^2(0,\tau)}\Big) + \|k_1' - k_2'\|_{L^2(0,\tau)}$$

$$\le C\Big(\tau\left(\|u_0\|_{H^2(I)}+\tau^{1/2}M\right)\|\tilde{k}_1'-\tilde{k}_2'\|_{L^2(0,\tau)}+\tau^{3/2}M\|(\tilde{v}_1-\tilde{v}_2)\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))}$$

$$+(p+q\tau)\Big\{\frac{\tau^{1/2}}{|\psi(l)|}\|\psi\|_{L^2(I)}\left(\tau^{1/2}+|k(0)|\tau^{3/2}+M\right)\|(\tilde{v}_1-\tilde{v}_2\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))}$$

$$+\frac{\tau^{1/2}}{|\psi(l)|}\left(\|f'\|_{L^2(0,\tau)}+M\|\psi\|_{L^2(I)}\right)\|\tilde{k}_1'-\tilde{k}_2'\|_{L^2(0,\tau)}\Big\}\Big)$$

$$+\left(1+C(p+q\tau)|G[u_{xx}](0)\right)|\|k_1'-k_2'\|_{L^2(0,\tau)}.$$

Using (4.17) finally we arrive at

$$\|v_1-v_2\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))}+\|k_1'-k_2'\|_{L^2(0,\tau)}\le$$

$$\tau^{1/2}\left\{\tau^{1/2}C\left(\|u_0\|_{H^2(I)}+\tau^{1/2}M\right)+\frac{p+q\tau}{\psi(l)}\left(\|f'\|_{L^2(0,\tau)}+M\|\psi\|_{L^2(I)}\right)\right.$$

$$\left.+\alpha M\left(1+C(p+q\tau)|G[u_{xx}](0)\right)\right\}\|\tilde{k}_1'-\tilde{k}_2'\|_{L^2(0,\tau)}$$

$$+\tau^{1/2}\left\{C\tau M+C\frac{p+q\tau}{\psi(l)}\|\psi\|_{L^2(I)}\left(\tau^{1/2}+|k(0)|\tau^{3/2}+M\right)\right.$$

$$\left.+\alpha\|\varphi'''\|_{L^2(I)}\left(1+C(p+q\tau)|G[u_{xx}](0)\right)\left(\tau^{1/2}+|k(0)|\tau^{3/2}+\tau^2M\right)\right\}\times$$

$$\times\|(\tilde{v}_1-\tilde{v}_2\|_{H^2(0,\tau;H_0^1(I)\cap H^2(I))}.$$

If we take $\tau>0$ small enough, then we have $A:Q(\tau,M)\to Q(\tau,M)$ is a contraction mapping. Thus, from the contraction mapping theorem, one can conclude that for a small time τ, there exists a unique solution $(v,k')\in H^3(0,\tau;H_0^1(I)\cap H^2(I))\times L^2(0,\tau)$ to the equivalent system (3.1)-(3.6) in $[0,\tau]$, and the proof is completed.

5 Conclusion

In the work, the local unique solvability of a kernel identification problem for a wave equation of memory type with acoustic boundary conditions was investigated. The considered problem was reduced to equivalent problem for a system integro-defferential equations with homogeneous boundary conditions. Its equivalence to the original problem is shown. Then using the contraction mappings principle in Sobolev space, the existence and uniqueness theorem for the equivalent problem is proved. Further, it is proposed to extend this method to the multidimensional case of domain with $I\subset\mathbb{R}^n$ $(n\ge 1)$ and investigate unique global solvability.

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Totieva Z. D.
Southern Mathematical Institute of Vladikavkaz Scientific Center RAS,
53, Vatutina str.,
362025, Vladikavkaz, Russia,
North-Caucasus Center for Mathematical Research of the Vladikavkaz Scientific Centre RAS,
1, Williams str.,
1363110, Mikhailovskoe, North Ossetia-Alania, Russia,
e-mail: jannatuaeva@inbox.ru

Durdiev D. K.
Bukhara Branch of the V. I. Romanovskiy Institute of Mathematics Academy of Sciences of the Republic of Uzbekistan,
11, M. Ikbal,
200117, Bukhara, Uzbekistan
d.durdiev@mathinst.uz

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