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AN ONE-DIMENSIONAL INVERSE PROBLEM FOR THE WAVE EQUATION

Romanov V. G., Bugueva T. V.

Abstract For the wave equation with inhomogeneity $\sigma(x)u_t^m + q(x)u^p$ a forward and an one-dimensional inverse problems are studied. Here $m > 1$ and $p > 1$ are real numbers. The forward problem is considered in the domain $x > 0, t > 0$ with zero initial data and Dirichlet boundary condition at $x = 0$. An unique solvability theorem of this problem is proved. The inverse problem is devoted to determining the coefficients $\sigma(x)$ and $q(x)$. As an additional information for recovering this coefficients, two forward problems with different Dirichlet data are considered and traces of the derivative of their solutions with respect to x are given at $x = 0$ on a finite interval. For the inverse problem a local existence and uniqueness theorem is established.

Key words: nonlinear wave equation, inverse problem, existence, uniqueness.

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1 Introduction

In recent years, there has been an increasing number of scientific papers devoted to solving both forward and inverse problems for nonlinear wave equations. Equations containing nonlinearities of the form $|u|^{p-1}u$ are called defocusing. For example, various formulations of forward problems and methods for solving them are considered in papers [1–6]. Thus, in the paper [1], the internal stabilization and control of the critical nonlinear Klein–Gordon equation $\Box u+u+|u|^4u = g$ on 3-D compact manifolds are studied. In paper [2], the authors prove the exponential stabilization of the semilinear wave equation $\Box u = \gamma(x)\partial_t u + \beta u + f(u)$, with an effective damping in a zone satisfying a geometric control condition only. The nonlinearity is assumed to be subcritical, defocusing and analytic. In [3], the global behaviors of solutions to defocusing semilinear wave equation $\Box\phi = |\phi|^{p-1}\phi$ in $\mathbb{R}^{1+d}$, $d \geqslant 3$, is investigated. For the case $p > 1 + 2/(d-1)$, an uniform weighted energy estimate for the solution is obtained, as well as an inverse polynomial attenuation of the energy flow through hypersurfaces away from the light cone is found. In [4], a wave equation $\Box u+|u|^{p-1}u = 0$ with a power nonlinearity is considered, defined outside the unit ball in $\mathbb{R}^n$, $n \geqslant 3$, with Dirichlet boundary conditions. It is proved that if $p > n + 3$ and the initial data are nonradial perturbations of large radial data, then there exists a global smooth solution. The solution is unique in the energy class solutions satisfying an energy inequality. Paper [5] is devoted to the study of the asymptotic behavior of solutions to the one-dimensional wave equation $\Box u + |u|^{p-1}u = 0$. It is proved that the solution with finite energy tends to zero in the pointwise sense, moreover, for sufficiently localized data belonging to some weighted energy space, the solution decays in time with inverse polynomial velocity. In [6], the equation $\Box\phi + |\phi|^{p-1}\phi = 0$ is studied on $\mathbb{R}\times\mathbb{R}^2\setminus\mathcal{K}$ with the Dirichlet boundary condition. Here, $\mathcal{K}$ is is a star-shaped obstacle with smooth boundary. It is proven that the solution scatters both in energy space and the critical Sobolev space.

Inverse problems for nonlinear wave equations have been studied relatively recently, but many results have already been obtained in solving these problems. Thus, in [7–9] various formulations of inverse problems related to the determination of the Lorentz metric or the coefficients included in these equations are considered. In [7], nonlinear inverse problems for the wave equation $\Box_g u(x) + H(x, u(x))$ are considered on a Lorentzian manifold M with Laplacian—Beltrami operator. It is shown that, on a given space-time (M, g), the source-to-solution map determines some coefficients of the Taylor expansion of H in u. In [8], for the semilinear wave equation $\Box_g u + w(x, u, \nabla_g u) = 0$ on Lorentzian manifolds, the inverse problem of determining the background Lorentzian metric is studied. In [9], the inverse boundary value problem is considered for a semilinear wave equation $\Box u + H(x, u(x)) = 0$ on a time-dependent Lorentzian manifold $\mathcal{M}$, with a time-like boundary. It is assumed that $H(x, z) \sim \sum\limits_{k=2}^{\infty} h_k(x)z^k$, where $h_k \in C^\infty(\mathcal{M})$. The time-dependent coefficients in the nonlinear terms of the equation can be reconstructed using knowledge of the Neumann-Dirichlet mapping, which allows for the reconstruction of the time-dependent terms. It was shown that either distorted plane waves or Gaussian beams can be used to derive uniqueness. In [10], the inverse problem of recovering the nonlinearity $f(x, u)$ in the differential equation $\Box u + f(x, u) = 0$ is considered. It is demonstrated that it is possible to recover the function $f(x, u)$ when it is odd in u, and it is also possible to recover the function $\alpha(x)$ when $f(x, u) = \alpha(x)u^{2m}$. In [11], the geometric non-linear inverse problem of recovering a Hermitian connection A from the source-to-solution map of the cubic wave equation $\Box_A u + \kappa|u|^2u = f$, is considered. Here $\kappa \neq 0$, $\Box_A$ is the connection wave operator in Minkowski space $\mathbb{R}^{1+3}$. The microlocal analysis is used for this nonlinear wave interactions. In [12], it is shown that the scattering operator for defocusing energy critical semilinear wave equations $\Box u + f(u) = 0$, $f \in C^\infty(\mathbb{R})$, $f \sim u^5$, defines the function f. In [13], the recovery of a potential associated with a semi-linear wave equation $\Box u + au^m = 0$ in $\mathbb{R}^{n+1}$, $n \geqslant 1$, is investigated, where m is integer number, $m \geqslant 2$. The Hölder stability estimate for the recovery of an unknown potential $a(x, t)$ from its Dirichlet-to-Neumann map is proved. In [14] the equation $\Box u + \alpha(x)|u|^2u = 0$ is considered in two-dimensional and three-dimensional spaces. The inverse problems of restoring the function $\alpha(x)$, $0 \leq \alpha(x) \in C_0^\infty$ are investigated, and it is shown that using the Radon transform, an unknown coefficient can be restored. In [15] the inverse problem of nonlinear ultrasound imaging analysis is considered. The propagation of ultrasonic waves is modeled by a quasi-linear wave equation. By making measurements at the boundary of the medium encoded in the Dirichlet—Neumann mapping, the nonlinearity is restored. In [16] the several inverse problems related to nonlinear progressive waves that occur during infrasound inversions are investigated. The nonlinear progressive equation has the quasi-linear form $u_{tt} = \Delta f(x, u)$, where

$f(x,u) = c_1(x)u + c_2(x)u^n$, $n \geqslant 2$, and can be recovered from the hyperbolic system of conservation laws associated with the Euler equation of the original equation. Unique identification results were obtained in determining $f(x,u)$, as well as related source data by measuring boundaries. The analysis of the problems is based on high-order linearization and the construction of Gaussian ray solutions for wave equations. In [17], for the nonlinear partial differential equation $\square u = q(x)u^{\gamma+1}$, where $\gamma > 0$, the inverse problem of determining the function $q(x)$ from boundary data is considered. Here, it is assumed that the desired function q is a continuous and finite function for $x \in \mathbb{R}^3$. It is shown that solutions to the corresponding forward problem for the given differential equation are bounded in some neighborhood of the characteristic curve, and an asymptotic expansion for the solution in this neighborhood is obtained. A theorem on the uniqueness of solutions to the inverse problem is proved. In [18] the equation $\square u = q(t)u_x^m$, where $m > 1$ is a real number, is considered. Theorems of the existence and uniqueness of the solution of the forward problem and a local existence and stability of the solution of the inverse problem are proved. In [19] an one-dimensional inverse problem of determining the nonlinear coefficient for a second-order hyperbolic equation with nonlinear absorption: $\square u + \sigma(x)|u_t|^m u_t = 0$, is studied, here $m > 0$ is a real number. For the inverse problem, a local existence and uniqueness theorem and a global stability estimate of its solutions are stated. In paper [20] a hyperbolic equation with variable leading part and nonlinearity in the lower order term is considered. The coefficients of the equation are smooth functions constant beyond some compact domain in the three-dimensional space. A plane wave with direction ℓ falls to the heterogeneity from the exterior of this domain. A solution to the corresponding Cauchy problem for the original equation is measured at boundary points of the domain for a time interval including the moment of arrival of the wave at these points. The unit vector ℓ is assumed to be a parameter of the problem and can run through all possible values sequentially. The inverse problem of determining the coefficient of the nonlinearity on using this information about solutions is studied. The structure of a solution to the direct problem and demonstrate that the inverse problem reduces to an integral geometry problem is described. The latter problem consists of constructing the desired function on using given integrals of the product of this function and a weight function. The integrals are taken along the geodesic lines of the Riemannian metric associated with the leading part of the differential equation. This new problem is analyzed and some estimate of the stability of its solution is found, which gives an estimate of the stability of solutions to the inverse problem.

In the present paper we consider an one-dimensional inverse problem for equation $\square u - \sigma(x)u_t^m - q(x)u^p = 0$ on semi-axis $x > 0$ with zero initial data and the boundary condition $u(0,t) = f(t)$, $m > 1$, $p > 1$ are a real number. The main goal is to recover coefficients $\sigma(x)$, $q(x)$ from the derivative $u_x(0,t)$ given for $t \in [0,T]$. We prove an uniqueness and existence theorem for the forward problem when the function $f(t)$ is given. Then we study the inverse problem and state a local uniqueness and existence theorem for this problem. Both theorems for forward and inverse problems are new in the theory of inverse problems.

2 Posing of problems

Let T be a real positive number.

A forward problem. Determine the function $u(x,t)$ satisfying the relations

$$u_{tt} - u_{xx} - \sigma(x)u_t^m - q(x)u^p = 0, \quad x > 0, \quad t \in (0,T]; \tag{1}$$

$$u|_{t=0} = u_t|_{t=0} = 0, \tag{2}$$

$$u|_{x=0} = f(t), \tag{3}$$

where $\sigma(x)$ and $q(x)$ are continuous functions; $m > 1$ and $p > 1$ are real numbers; $f(t)$ is the twice continuously differentiable function and $f(0) = a > 0$, $f'(0) = b > 0$; a, b are some constants.

An inverse problem. Let $f_k(t)$, $k = 1,2$, be the given functions and $f_k(0) = a_k > 0$, $f_k'(0) = b_k > 0$, $k = 1,2$, numbers a_k and b_k satisfy the condition $a_1^p b_2^m - a_2^p b_1^m > 0$. The solution of the forward problem (1)–(3) for $f = f_k$, $k = 1,2$, denote $u_k(x,t)$, $k = 1,2$. Find the functions $\sigma(x)$ and $q(x)$ from the given information about solutions $u_k(x,t)$:

$$(u_k)_x|_{x=0} = h_k(t), \quad t \in [0,T], \quad k = 1,2. \tag{4}$$

3 An analysis of the forward problem

The solution of the problem (1)–(3) can be written as

$$u(x,t) = f(t-x) + \frac{1}{2}\int_0^t d\tau \int_{|x-t+\tau|}^{x+t-\tau} \left[\sigma(\xi)u_t^m(\xi,\tau) + q(\xi)u^p(\xi,\tau)\right] d\xi. \tag{5}$$

Since $u = 0$ for $t < x$, equation (5) takes the form

$$u(x,t) = f(t-x) + \frac{1}{2}\iint\limits_{D_1(x,t)} \left[\sigma(\xi)u_t^m(\xi,\tau) + q(\xi)u^p(\xi,\tau)\right] d\xi d\tau, \tag{6}$$

where the domain $D_1(x,t)$ is a rectangle bounded by the characteristics

$$\xi + \tau = t + x, \quad \xi + \tau = t - x, \quad \xi - \tau = x - t, \quad \xi - \tau = 0.$$

Let $G(T) = \{(x,t) \mid 0 \leqslant x < t \leqslant T - x\}$ and $(x,t) \in G(T)$. Rewrite equation (4) as a sum of repeated integrals

$$\begin{aligned} u(x,t) = f(t-x) &+ \frac{1}{2}\int_0^{(t-x)/2} d\xi \int_{t-x-\xi}^{t-x+\xi} \left[\sigma(\xi)u_t^m(\xi,\tau) + q(\xi)u^p(\xi,\tau)\right] d\tau \\ &+ \frac{1}{2}\int_{(t-x)/2}^{x} d\xi \int_{\xi}^{t-x+\xi} \left[\sigma(\xi)u_t^m(\xi,\tau) + q(\xi)u^p(\xi,\tau)\right] d\tau \\ &+ \frac{1}{2}\int_{x}^{(x+t)/2} d\xi \int_{\xi}^{t+x-\xi} \left[\sigma(\xi)u_t^m(\xi,\tau) + q(\xi)u^p(\xi,\tau)\right] d\tau. \end{aligned} \tag{7}$$

Differentiating (7), we find

$$u_t(x,t) = f'(t-x) + \frac{1}{2}\int\limits_0^x [\sigma(\xi)u_t^m(\xi, t-x+\xi) + q(\xi)u^p(\xi, t-x+\xi)]\, d\xi$$
$$+ \frac{1}{2}\int\limits_x^{(x+t)/2} [\sigma(\xi)u_t^m(\xi, x+t-\xi) + q(\xi)u^p(\xi, x+t-\xi)]\, d\xi$$
$$- \frac{1}{2}\int\limits_0^{(t-x)/2} [\sigma(\xi)u_t^m(\xi, t-x-\xi) + q(\xi)u^p(\xi, t-x-\xi)]\, d\xi. \quad (8)$$

Definition 3.1. Let's say that $(\sigma(x), q(x)) \in \mathcal{S}_0(\kappa_0)$ if $\sigma \in C[0, T/2]$, $q \in C[0, T/2]$ and they satisfy the conditions

$$\max\{|\sigma(x)|, |q(x)|\} \leqslant \kappa_0, \quad x \in [0, T/2], \quad (9)$$

with a constant $\kappa_0 > 0$.

Lemma 3.1. *Let $\mu_0 = \min\{a, b\} < 2/3$, $\gamma = \min\{m, p\}$, $(\sigma(x), q(x)) \in \mathcal{S}_0(\kappa_0)$ and the inequalities*

$$\begin{gathered} 0 < a \leqslant f(t) \leqslant F_1 \leq 1 - \mu_0/2, \quad 0 < b \leqslant f'(t) \leqslant F_1 \leq 1 - \mu_0/2, \\ 1 - (\gamma - 1)F_1^{\gamma-1}\omega T \geqslant (1 - \mu_0/2)^{\gamma-1} \end{gathered} \quad (10)$$

are fulfilled. Here $\omega = \omega(\kappa_0, T) = \kappa_0 \max\{3, T/2\}$. Then there is a unique continuous solution in $G(T)$ of equations (7) and (8) and the folloring estimates hold

$$\begin{gathered} 0 < \mu_0/2 \leqslant u(x,t) \leqslant \frac{F_1}{\left[1 - (\gamma-1)F_1^{\gamma-1}\omega t\right]^{1/(\gamma-1)}} \leqslant 1, \\ 0 < \mu_0/2 \leqslant u_t(x,t) \leqslant \frac{F_1}{\left[1 - (\gamma-1)F_1^{\gamma-1}\omega t\right]^{1/(\gamma-1)}} \leqslant 1, \quad (x,t) \in G(T). \end{gathered} \quad (11)$$

Remark 3.1. For any fixed $m > 1$, $p > 1$, κ_0 and $T > 0$, one can always choose the function $f(t)$ and the numbers a and b such that F_1 be much less than 1 and second condition (10) be fulfilled. If the numbers a and b are fixed, this condition is a smallness condition for T or κ_0.

Proof. Let's change the integration variable ξ by τ in (8). Then this equation will be written as

$$u_t(x,t) = f'(t-x) + \frac{1}{2}\int\limits_{t-x}^{t} \left[\sigma(x-t+\tau)u_t^m(x-t+\tau, \tau) + q(x-t+\tau)u^p(x-t+\tau, \tau)\right] d\tau$$

$$+\frac{1}{2}\int\limits_{(x+t)/2}^{t}\big[\sigma(x+t-\tau)u_t^m(x+t-\tau,\tau)+q(x+t-\tau)u^p(x+t-\tau,\tau)\big]\,d\tau$$

$$-\frac{1}{2}\int\limits_{(t-x)/2}^{t-x}\big[\sigma(t-x-\tau)u_t^m(t-x-\tau,\tau)+q(t-x-\tau)u^p(t-x-\tau,\tau)\big]\,d\tau. \quad (12)$$

Assuming the continuity of the functions $u(x,t)$ and $u_t(x,t)$ (that will be proved later), introduce the new function

$$z(t)=\begin{cases}\max\Big\{\max\limits_{0\leqslant x\leqslant t}|u(x,t)|,\max\limits_{0\leqslant x\leqslant t}|u_t(x,t)|\Big\}, & t\in[0,T/2],\\ \max\Big\{\max\limits_{0\leqslant x\leqslant T-t}|u(x,t)|\max\limits_{0\leqslant x\leqslant T-t}|u_t(x,t)|\Big\}, & t\in[T/2,T].\end{cases} \quad (13)$$

It follows from equations (6) and (12) and conditions (11) that $z(t)\le 1$, at least for t sufficiently close to zero (in fact it is right for all $t\in[0,T]$). Using this argument, we replace z^m and z^p in the estimates below with z^γ, $\gamma=\min(m,p)$. Then the following inequalities follow from (6) and (12):

$$|u(x,t)|\leqslant F_1+\frac{\kappa_0 T}{2}\int\limits_0^t z^\gamma(\tau)\,d\tau, \quad |u_t(x,t)|\leqslant F_1+3\kappa_0\int\limits_0^t z^\gamma(\tau)\,d\tau.$$

Therefore,

$$z(t)\leqslant F_1+\omega\int\limits_0^t z^\gamma(\tau)\,d\tau, \quad \omega=\kappa_0\max(3,T/2). \quad (14)$$

Let's denote

$$z_1(t)=F_1+\omega\int\limits_0^t z^\gamma(\tau)\,d\tau. \quad (15)$$

By virtue of (14), (15) we have

$$\frac{dz_1(t)}{dt}=\omega z^\gamma(t)\leqslant\omega z_1^\gamma(t), \quad z_1(0)=F_1.$$

Integrating the resulting inequality, we find that

$$z_1^{1-\gamma}(t)\geqslant F_1^{1-\gamma}+(1-\gamma)\omega t,$$

or

$$z_1^{\gamma-1}(t)\leqslant\frac{F_1^{\gamma-1}}{[F_1^{1-\gamma}+(1-\gamma)\omega t]}\leqslant\frac{F_1^{\gamma-1}}{[1-(\gamma-1)F_1^{\gamma-1}\omega T]}\leqslant\left(\frac{F_1}{1-\mu_0/2}\right)^{\gamma-1}\leqslant 1, \quad t\in[0,T]. \quad (16)$$

Here final inequalities follow from conditions (10). Hence,

$$z(t) \leqslant z_1(t) \leqslant \frac{F_1}{[1-(\gamma-1)F_1^{m-1}\omega t]^{1/(\gamma-1)}} \leqslant 1, \quad t \in [0,T]. \tag{17}$$

On the other hand, by virtue of (17), it follows from equations (6), (12) that

$$\begin{aligned}
\min\{u(x,t), u_t(x,t)\} &\geqslant \mu_0 - \omega \int_0^t z^\gamma(\tau)\,d\tau \\
&\geqslant \mu_0 - \omega \int_0^t \frac{F_1^\gamma}{[1-(\gamma-1)F_1^{\gamma-1}\omega\tau]^{\gamma/(\gamma-1)}}\,d\tau \\
&= \mu_0 + \frac{F_1}{(\gamma-1)} \int_0^t \frac{d[1-(\gamma-1)F_1^{\gamma-1}\omega\tau]}{[1-(\gamma-1)F_1^{\gamma-1}\omega\tau]^{\gamma/(\gamma-1)}} \\
&= \mu_0 + \frac{F_1}{(\gamma-1)} \frac{-(\gamma-1)}{[1-(\gamma-1)F_1^{\gamma-1}\omega\tau]^{1/(\gamma-1)}}\bigg|_0^t \\
&= \mu_0 - F_1\left[\frac{1}{[1-(\gamma-1)F_1^{\gamma-1}\omega t]^{1/(\gamma-1)}} - 1\right] \\
&\geqslant \mu_0 - \left(1-\frac{\mu_0}{2}\right)\left(\frac{2}{2-\mu_0}-1\right) = \frac{\mu_0}{2}.
\end{aligned} \tag{18}$$

The estimate (11) follows from (17), (18).

Let us now prove that under the condition (10) there exists a continuous in $G(T)$ solution of the equations (7) and (12). Denote $v(x,t) := u_t(x,t)$ and define successive approximations, assuming that

$$u_0(x,t) = f(t-x), \quad v_0(x,t) = f'(t-x),$$

$$u_n(x,t) = f(t-x) + \frac{1}{2}\iint\limits_{D_1(x,t)} \left[\sigma(\xi)v_{n-1}^m(\xi,\tau) + q(\xi)u_{n-1}^p(\xi,\tau)\right] d\xi d\tau,$$

$$\begin{aligned}
v_n(x,t) = f'(t-x) &+ \frac{1}{2}\int_{t-x}^t \left[\sigma(x-t+\tau)v_{n-1}^m(x-t+\tau,\tau) + q(x-t+\tau)u_{n-1}^p(x-t+\tau,\tau)\right] d\tau \\
&+ \frac{1}{2}\int_{(x+t)/2}^t \left[\sigma(x+t-\tau)v_{n-1}^m(x+t-\tau,\tau) + q(x+t-\tau)u_{n-1}^p(x+t-\tau,\tau)\right] d\tau \\
&- \frac{1}{2}\int_{(t-x)/2}^{t-x} \left[\sigma(t-x-\tau)v_{n-1}^m(t-x-\tau,\tau) + q(t-x-\tau)u_{n-1}^p(t-x-\tau,\tau)\right] d\tau,
\end{aligned}$$

$$n = 1,2,\ldots, \quad (x,t) \in G(T). \tag{19}$$

Let's check that for $(x,t)\in G(T)$ the inequalities are valid

$$\begin{gathered}\mu_0/2\leqslant u_n(x,t)\leqslant\frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}}\leqslant 1,\\ \mu_0/2\leqslant v_n(x,t)\leqslant\frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}}\leqslant 1,\quad n=0,1,\dots.\end{gathered}\tag{20}$$

It is convenient to use designations like (13) for this:

$$z_n(t)=\begin{cases}\max\Big\{\max\limits_{0\leqslant x\leqslant t}|u_n(x,t)|,\max\limits_{0\leqslant x\leqslant t}|v_n(x,t)|\Big\}, & t\in[0,T/2],\\ \max\Big\{\max\limits_{0\leqslant x\leqslant T-t}|u_n(x,t)|\max\limits_{0\leqslant x\leqslant T-t}|v_n(x,t)|\Big\}, & t\in[T/2,T],\quad n=0,1,2,\dots.\end{cases}$$

For $n=0$ inequalities

$$\mu_0/2\leqslant z_0(t)\leqslant F_1\leqslant\frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}}\leqslant 1,\quad t\in[0,T],\tag{21}$$

follow from (10) and the condition $\gamma>1$.

Using the method of mathematical induction, assume that for all $1\leqslant n\leqslant k$ the inequalities hold

$$\mu_0/2\leqslant z_n(x,t)\leqslant\frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}}\leqslant 1,\quad t\in[0,T].$$

Then

$$\begin{aligned}z_{k+1}(x,t)&\leqslant F_1+\omega\int_0^t z_k^\gamma(\tau)\,d\tau\leqslant F_1+\omega\int_0^t\frac{F_1^\gamma\,d\tau}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega\tau\big]^{\gamma/(\gamma-1)}}\\ &\leqslant F_1+F_1\left[\frac{1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}}-1\right]\\ &=\frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}}\leqslant 1,\quad t\in[0,T].\end{aligned}\tag{22}$$

On the other hand,

$$\begin{aligned}z_{k+1}(x,t)&\geqslant\mu_0-\omega\int_0^t z_k^\gamma(\tau)\,d\tau\geqslant\mu_0-\omega\int_0^t\frac{F_1^\gamma}{[1-(\gamma-1)F_1^{\gamma-1}\omega\tau]^{m/(m-1)}}\,d\tau\\ &=\mu_0-F_1\left[\frac{1}{[1-(\gamma-1)F_1^{\gamma-1}\omega t]^{1/(m-1)}}-1\right]\\ &\geqslant\mu_0-\Big(1-\frac{\mu_0}{2}\Big)\Big(\frac{2}{2-\mu_0}-1\Big)=\frac{\mu_0}{2},\quad t\in[0,T].\end{aligned}\tag{23}$$

By virtue of the method of mathematical induction, the validity of inequalities (20) follows from (21)–(23).

Consider the differences

$$\overline{u}_n(x,t) = u_n(x,t) - u_{n-1}(x,t),\ \overline{v}_n(x,t) = v_n(x,t) - v_{n-1}(x,t), \quad n = 1,2,\dots,$$

and define

$$\overline{z}_n(t) = \begin{cases} \max\Big\{ \max\limits_{0\leqslant x\leqslant t} |\overline{u}_n(x,t)|, \max\limits_{0\leqslant x\leqslant t} |\overline{v}_n(x,t)| \Big\}, & t \in [0, T/2], \\ \max\Big\{ \max\limits_{0\leqslant x\leqslant T-t} |\overline{u}_n(x,t)| \max\limits_{0\leqslant x\leqslant T-t} |\overline{v}_n(x,t)| \Big\}, & t \in [T/2, T],\ n = 1,2,\dots. \end{cases}$$

From (19) we have

$$\overline{u}_1(x,t) = \frac{1}{2} \iint\limits_{D_1(x,t)} \big[\sigma(\xi) v_0^m(\xi,\tau) + q(\xi) u_0^p(\xi,\tau)\big]\, d\xi d\tau,$$

$$\overline{v}_1(x,t) = \frac{1}{2} \int\limits_{t-x}^{t} \big[\sigma(x-t+\tau) v_0^m(x-t+\tau,\tau) + q(x-t+\tau) u_0^p(x-t+\tau,\tau)\big]\, d\tau$$

$$+ \frac{1}{2} \int\limits_{(x+t)/2}^{t} \big[\sigma(x+t-\tau) v_0^m(x+t-\tau,\tau) + q(x+t-\tau) u_0^p(x+t-\tau,\tau)\big]\, d\tau$$

$$- \frac{1}{2} \int\limits_{(t-x)/2}^{t-x} \big[\sigma(t-x-\tau) v_0^m(t-x-\tau,\tau) + q(t-x-\tau) u_0^p(t-x-\tau,\tau)\big]\, d\tau, \quad (24)$$

$$\overline{u}_{n+1}(x,t) = \frac{1}{2} \iint\limits_{D_1(x,t)} \Big\{\sigma(\xi)\big[v_n^m(\xi,\tau) - v_{n-1}^m(\xi,\tau)\big] + q(\xi)[u_n^p(\xi,\tau) - u_{n-1}^p(\xi,\tau)]\Big\}\, d\xi d\tau,$$

$$\overline{v}_{n+1}(x,t) = \frac{1}{2} \int\limits_{t-x}^{t} \Big\{\sigma(x-t+\tau)\big[v_n^m(x-t+\tau,\tau) - v_{n-1}^m(x-t+\tau,\tau)\big]$$
$$+ q(x-t+\tau)\big[u_n^p(x-t+\tau,\tau) - u_{n-1}^p(x-t+\tau,\tau)\big]\Big\}\, d\tau$$
$$+ \frac{1}{2} \int\limits_{(x+t)/2}^{t} \Big\{\sigma(x+t-\tau)\big[v_n^m(x+t-\tau,\tau) - v_{n-1}^m(x+t-\tau,\tau)\big]$$
$$+ q(x+t-\tau)\big[u_n^p(x+t-\tau,\tau) - u_{n-1}^p(x+t-\tau,\tau)\big]\Big\}\, d\tau$$
$$- \frac{1}{2} \int\limits_{(t-x)/2}^{t-x} \Big\{\sigma(t-x-\tau)\big[v_n^m(t-x-\tau,\tau) - v_{n-1}^m(t-x-\tau,\tau)\big]$$
$$+ q(t-x-\tau)\big[u_n^p(t-x-\tau,\tau) - u_{n-1}^p(t-x-\tau,\tau)\big]\Big\}\, d\tau,$$

$$n = 1, 2, \dots, \quad (x,t) \in G(T). \quad (25)$$

From the equalities (24) we find the estimates

$$|\overline{u}_1(x,t)| \leqslant \frac{\kappa_0 T}{2} \int\limits_0^t d\tau = \frac{\kappa_0 T}{2} t, \quad |\overline{v}_1(x,t)| \leqslant 3\kappa_0 \int\limits_0^t d\tau = 3\kappa_0 t.$$

Therefore,

$$\overline{z}_1(t) \leqslant \omega t. \quad (26)$$

Let's imagine the difference $u_n^p(\xi,\tau) - u_{n-1}^p(\xi,\tau)$ as:

$$u_n^p(\xi,\tau) - u_{n-1}^p(\xi,\tau) = p \int\limits_{u_{n-1}(\xi,\tau)}^{u_n(\xi,\tau)} r^{p-1}\, dr = \overline{u}_n(\xi,\tau) K_{n,p}[u_n, u_{n-1}](\xi,\tau), \quad (27)$$

where

$$K_{n,p}[u_n, u_{n-1}](\xi,\tau) = p \int\limits_0^1 [s u_n(\xi,\tau) + (1-s) u_{n-1}(\xi,\tau)]^{p-1}\, ds. \quad (28)$$

Then in full force (20), (27), (28) inequalities

$$\begin{gathered} |q(\xi) K_{n,p}[u_n, u_{n-1}](\xi,\tau)| \leqslant p\kappa_0, \\ |\sigma(\xi) K_{n,m}[v_n, v_{n-1}](\xi,\tau)| \leqslant m\kappa_0, \quad n = 1, 2, \dots, \quad (\xi,\tau) \in G(T), \end{gathered} \quad (29)$$

are true.

Denote $\gamma_1 = \max\{m, p\}$. Then, using inequalities (29), we find that

$$|\overline{u}_{n+1}(x,t)| \leqslant \frac{\kappa_0 \gamma_1 T}{2} \int\limits_0^t \overline{z}_n(\tau)\, d\tau,$$

$$|\overline{v}_{n+1}(x,t)| \leqslant 3\kappa_0 \gamma_1 \int\limits_0^t \overline{z}_n(\tau)\, d\tau, \quad (x,t) \in G(T).$$

Hence,

$$\overline{z}_{n+1}(t) \leqslant \omega \gamma_1 \int\limits_0^t \overline{z}_n(\tau)\, d\tau, \quad t \in [0,T]. \quad (30)$$

Inequalities (26), (30) are followed by easily verifiable estimates

$$\overline{z}_n(t) \leqslant (\omega\gamma_1)^{n-1} \frac{\omega t^n}{n!} \leqslant (\omega\gamma_1)^{n-1} \frac{\omega T^n}{n!}, \quad n = 1, 2, \dots, \quad t \in [0,T]. \quad (31)$$

It follows from estimate (31) that the sequences of continuous functions $u_n(x,t)$ and $v_n(x,t)$ converge uniformly in the domain $G(T)$ and determine the continuous solution of the equations (6) and (8). Inequalities (11) are fulfilled for this solution. The uniqueness of the solution is established by the standard method.

Lemma 3.1 is proved. □

Differentiating (7) with respect to x, we obtain

$$u_x(x,t) = -f'(t-x) - \frac{1}{2}\int_0^x \big[\sigma(\xi)u_t^m(\xi,t-x+\xi) + q(\xi)u^p(\xi,t-x+\xi)\big]\,d\xi$$
$$+\frac{1}{2}\int_0^{(t-x)/2} \big[\sigma(\xi)u_t^m(\xi,t-x-\xi) + q(\xi)u^p(\xi,t-x-\xi)\big]\,d\xi$$
$$+\frac{1}{2}\int_x^{(x+t)/2} \big[\sigma(\xi)u_t^m(\xi,x+t-\xi) + q(\xi)u^p(\xi,x+t-\xi)\big]\,d\xi. \quad (32)$$

Since the expressions on the right hand side of equality (32) are continuous functions, then the expression on the left hand side is also a continuous function in the domain $G(T)$. Thus, the function $u_x \in C(G(T))$.

Lemma 3.2. *Under the assumptions of Lemma* 3.1, *the function* $u_x \in C(G(T))$ *and the following relations take place:*

$$u_x(0,0) = -f'(0) = -b, \quad (33)$$

$$\begin{gathered}\mu_0/2 \leqslant -u_x(x,t) \leqslant \frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}} \leqslant 1, \quad (x,t)\in G(T),\\ u_x(0,t)+f'(t) \geqslant -\mu_0/2, \quad f'(t)-u_x(0,t) \geqslant 3\mu_0/2, \quad t\in[0,T],\end{gathered} \quad (34)$$

$$\begin{gathered}f(t) + \int_0^t u_x(0,\tau)\,d\tau \geqslant \mu_0/2,\\ f(t) - \int_0^t u_x(0,\tau)\,d\tau \geqslant \mu_0/2, \quad t\in[0,T].\end{gathered} \quad (35)$$

Proof. The first relation (33) follows directly from equation (32) for $x=t=0$, Further, inequalities follow from equation (32)

$$-u_x(x,t \leqslant F_1 + \omega\int_0^t z^\gamma(\tau)\,d\tau \leqslant F_1 + \omega\int_0^t \frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega\tau\big]^{1/(\gamma-1)}}\,d\tau$$
$$= \frac{F_1}{\big[1-(\gamma-1)F_1^{\gamma-1}\omega t\big]^{1/(\gamma-1)}} \leq 1, \quad (x,t)\in G(T),$$

$$|u_x(x,t)+f'(t-x)| \leqslant \omega\int_0^t z^\gamma(\tau)\,d\tau \leqslant \mu_0/2, \quad (x,t)\in G(T),$$

Therefore

$$-u_x(,t) = f'(t-x) - [u_x(x,t) + f'(t-x)] \geqslant \mu_0/2,$$
$$u_x(0,t) + f'(t) \geqslant -\mu_0/2,$$
$$f'(t) - u_x(0,t) \geqslant \mu_0 + \mu_0/2 = 3\mu_0/2, \quad (x,t) \in G(T).$$

The relations (34) follow from the above inequalities.

Calculating the integral on the left side of the first relation (35), we find

$$\left|\int_0^t \big[u_x(0,\tau) + f'(\tau)\big]\, d\tau\right| \leqslant \left|\int_0^t \int_0^{\tau/2} \big[\sigma(\xi) u_\tau^m(\xi,\tau-\xi) + q(\xi) u^p(\xi,\tau-\xi)\big]\, d\xi\, d\tau\right|$$
$$\leqslant \omega \int_0^t z^\gamma(\tau)\, d\tau \leqslant \mu_0/2, \quad t \in [0,T].$$

From here

$$f(t) + \int_0^t u_x(0,\tau)\, d\tau = f(0) + \int_0^t [u_x(0,\tau) + f'(\tau)]\, d\tau \geqslant f(0) - \mu_0/2 \geqslant \mu_0/2,$$

$$f(t) - \int_0^t u_x(0,\tau)\, d\tau = 2f(t) - f(0) - \int_0^t [u_x(0,\tau) + f'(\tau)]\, d\tau$$
$$\geqslant 2f(t) - f(0) - \mu_0/2 \geqslant \mu_0/2, \quad t \in [0,T].$$

Inequalities follow from these relations (35). Lemma 3.2 is proved. □

Differentiating equality (8) with respect to t and and taking into account that $u(x,x) = f(0) = a$, we obtain

$$u_{tt}(x,t) = f''(t-x) + \frac{1}{4}\sigma\left(\frac{x+t}{2}\right) u_t^m\left(\frac{x+t}{2}, \frac{x+t}{2}\right) - \frac{1}{4}\sigma\left(\frac{t-x}{2}\right) u_t^m\left(\frac{t-x}{2}, \frac{t-x}{2}\right)$$
$$+ \frac{a^p}{4} q\left(\frac{x+t}{2}\right) - \frac{a^p}{4} q\left(\frac{t-x}{2}\right)$$
$$+\frac{1}{2}\int_0^x \big[m\sigma(\xi) u_t^{m-1}(\xi,t-x+\xi) u_{tt}(\xi,t-x+\xi) + pq(\xi) u^{p-1}(\xi,t-x+\xi) u_t(\xi,t-x+\xi)\big]\, d\xi$$
$$+\frac{1}{2}\int_x^{(x+t)/2} \big[m\sigma(\xi) u_t^{m-1}(\xi,x+t-\xi) u_{tt}(\xi,x+t-\xi) + pq(\xi) u^{p-1}(\xi,x+t-\xi) u_t(\xi,x+t-\xi)\big]\, d\xi$$
$$-\frac{1}{2}\int_0^{(t-x)/2} \big[m\sigma(\xi) u_t^{m-1}(\xi,t-x-\xi) u_{tt}(\xi,t-x-\xi) + pq(\xi) u^{p-1}(\xi,t-x-\xi) u_t(\xi,t-x-\xi)\big]\, d\xi,$$
$$(x,t) \in G(T). \quad (36)$$

Lemma 3.3. *Let the conditions of Lemma 1 be fulfilled and let* $f \in C^2[0,T]$*. Then there is a unique continuous in* $G(T)$ *solution of the equation* (36) *and the following estimate holds*

$$|u_{tt}(x,t)| \leqslant K, \quad (x,t) \in G(T), \tag{37}$$

where constant K *depends on* $\|f''\|_{C[0,T]}$, T, κ_0, m, p *and* a.

Proof. Replace the integration variable ξ in (36) with τ. Then we get the equation

$$\begin{aligned} u_{tt}(x,t) = f''(t-x) + \frac{1}{4}\sigma\left(\frac{x+t}{2}\right)u_t^m\left(\frac{x+t}{2},\frac{x+t}{2}\right) - \frac{1}{4}\sigma\left(\frac{t-x}{2}\right)u_t^m\left(\frac{t-x}{2},\frac{t-x}{2}\right) \\ + \frac{a^p}{4}q\left(\frac{x+t}{2}\right) - \frac{a^p}{4}q\left(\frac{t-x}{2}\right) \\ + \frac{1}{2}\int\limits_{t-x}^{t}\big[m\sigma(x-t+\tau)u_t^{m-1}(x-t+\tau,\tau)u_{tt}(x-t+\tau,\tau) \\ + pq(x-t+\tau)u^{p-1}(x-t+\tau,\tau)u_t(x-t+\tau,\tau)\big]\,d\tau \\ + \frac{1}{2}\int\limits_{(x+t)/2}^{t}\big[m\sigma(x+t-\tau)u_t^{m-1}(x+t-\tau,\tau)u_{tt}(x+t-\tau,\tau) \\ + pq(x+t-\tau)u^{p-1}(x+t-\tau,\tau)u_t(x+t-\tau,\tau)\big]\,d\tau \\ - \frac{1}{2}\int\limits_{(t-x)/2}^{t-x}\big[m\sigma(t-x-\tau)u_t^{m-1}(t-x-\tau,\tau)u_{tt}(t-x-\tau,\tau) \\ + pq(t-x-\tau)u^{p-1}(t-x-\tau,\tau)u_t(t-x-\tau,\tau)\big]\,d\tau. \end{aligned} \tag{38}$$

Equation (38) for found functions $u(x,t)$, $u_t(x,t)$, $(x,t) \in G(T)$, is the Volterra integral equation for the variable t relative to $u_{tt}(x,t)$. The kernel of this equation and the free term are continuous in $G(T)$. Therefore, there exists a single continuous solution to this equation. Therefore, equation (36) defines $u_{tt}(x,t)$ as a continuous function in $G(T)$. Hence, the evaluation (37). Lemma 3.3 is proved. □

Differentiating equality (8) with respect to x, we obtain

$$\begin{aligned} u_{xt}(x,t) = -f''(t-x) + \frac{1}{4}\sigma\left(\frac{x+t}{2}\right)u_t^m\left(\frac{x+t}{2},\frac{x+t}{2}\right) + \frac{1}{4}\sigma\left(\frac{t-x}{2}\right)u_t^m\left(\frac{t-x}{2},\frac{t-x}{2}\right) \\ + \frac{a^p}{4}q\left(\frac{x+t}{2}\right) + \frac{a^p}{4}q\left(\frac{t-x}{2}\right) \\ - \frac{1}{2}\int\limits_0^x\big[m\sigma(\xi)u_t^{m-1}(\xi,t-x+\xi)u_{tt}(\xi,t-x+\xi) + pq(\xi)u^{p-1}(\xi,t-x+\xi)u_t(\xi,t-x+\xi)\big]\,d\xi \\ + \frac{1}{2}\int\limits_x^{(x+t)/2}\big[m\sigma(\xi)u_t^{m-1}(\xi,x+t-\xi)u_{tt}(\xi,x+t-\xi) + pq(\xi)u^{p-1}(\xi,x+t-\xi)u_t(\xi,x+t-\xi)\big]\,d\xi \end{aligned}$$

$$+\frac{1}{2}\int\limits_0^{(t-x)/2}\big[m\sigma(\xi)u_t^{m-1}(\xi,t-x-\xi)u_{tt}(\xi,t-x-\xi)+pq(\xi)u^{p-1}(\xi,t-x-\xi)u_t(\xi,t-x-\xi)\big]\,d\xi,$$
$$(x,t)\in G(T). \quad (39)$$

Since the expressions on the right hand side of equality (39) are continuous functions, then the expression on the left hand side is also a continuous function in the domain $G(T)$. The continuity of the function $u_{xx}(x,t)$ in the domain $G(T)$ follows from equation(1) and the above-proven continuity of the functions $u(x,t)$, $u_t(x,t)$ and $u_{tt}(x,t)$.

From Lemmas 3.1, 3.2, 3.3 and the above-established continuity of derivatives $u_{xt}(x,t)$ and $u_{xx}(x,t)$ follows

Theorem 3.1. *Forward problem* (1)–(3) *has unique solution and it is a function of the class* $C^2(G(T))$.

We present some corollaries from Lemma 3.2 and Theorem 3.1, those are useful for studying the inverse problem. Let's denote $h(t)=u_x(0,t)$.

Corollary 3.1. From the fact that the function $u(x,t)$ belongs to the space $C^2(G(T))$ it follows that the function $h(t)\in C^1[0,T]$, and from the relations (33)–(35) it follows that $h(t)$ has the following properties:

$$\begin{gathered}h(0)=-f'(0)=-b,\quad f'(t)+h(t)\geqslant -\mu_0/2,\quad f'(t)-h(t)\geqslant 3\mu_0/2,\\ f(t)+\int\limits_0^t h(\tau)\,d\tau\geqslant \mu_0/2,\quad f(t)-\int\limits_0^t h(\tau)\,d\tau\geqslant \mu_0/2,\quad t\in[0,T].\end{gathered} \quad (40)$$

Corollory 3.2. From (39) for $x=0$ the equality follows

$$h'(t)=-f''(t)+\frac{1}{2}\sigma(t/2)(u_t(t/2,t/2))^m+q(t/2)\frac{a^p}{2}$$
$$+\int\limits_0^{t/2}\big[m\sigma(\xi)u_t^{m-1}(\xi,t-\xi)u_{tt}(\xi,t-\xi)+pq(\xi)u^{m-1}(\xi,t-\xi)u_t(\xi,t-\xi)\big]\,d\xi. \quad (41)$$

We use an analogue of this equality in the next section.

An investigation of the inverse problem

Inverse problem. Let T be a given positive number, $f_k(t)$, $k=1,2$— given functions for $t\in[0,T]$ such that $f_k(0)=a_k>0$, $f_k'(0)=b_k>0$, and the numbers a_k and b_k satisfy the condition

$$D=a_1^p b_2^m-a_2^p b_1^m>0. \quad (42)$$

Let, in addition, from solutions of $u_k(x,t)$, $k=1,2$, of forward problem (1)–(3) for $f=f_k$, $k=1,2,$, the functions $h_k(t)$ defined by formula (4) are given. In the inverse problem, it is required to find functions $\sigma(x)$ and $q(x)$ from the given information.

Denote $\mu = \min\{a_1, b_1, a_2, b_2\}$.

Definition 4.1. Let's say that $(f_k(t), h_k(t)) \in \mathcal{F}(F, H, \mu)$ if $f_k \in C^2[0,T]$, $h_k \in C^1[0,T]$ and satisfy the conditions

$$h_k(0) = -f_k'(0) = -b_k, \quad k = 1,2, \tag{43}$$

and

$$\|f_k\|_{C^2[0,T]} \leqslant F, \quad \|h_k\|_{C^1[0,T]} \leqslant H, \quad k = 1,2, \tag{44}$$

$$\begin{gathered} f_k'(t) + h_k(t) \geqslant -\mu/2, \quad f_k'(t) - h_k(t) \geqslant 3\mu/2, \\ f_k(t) + \int_0^t h_k(\tau)\,d\tau \geqslant \mu/2, \quad f_k(t) - \int_0^t h_k(\tau)\,d\tau \geqslant \mu/2, \quad t \in [0,T]. \end{gathered} \tag{45}$$

$$\mu \leqslant a_k \leqslant f_k(t), \quad k = 1,2, \tag{46}$$

$$\mu \leqslant b_k \leqslant f_k'(t), \quad k = 1,2, \quad t \in [0,T], \tag{47}$$

where F, H — some given numbers.

Remark 4.1. Inequalities (45) are analogous of inequalities (40), which are obtained under the conditions of the lemma 3.1. Here they are simply postulated. The analysis of the forward problem shows that they are acceptable.

Theorem 4.2. *Let $f_k(t), h_k(t) \in \mathcal{F}(F, H, \mu)$. Then there exits a positive number $T_0 \leqslant T$ and a single pair of functions $\sigma \in C[0, T_0/2]$, $q \in C[0, T_0/2]$, such that the solutions of problem (1)–(3) for $f = f_k(t)$, $k = 1,2$, satisfy the condition (4) for $t \in [0, T_0]$.*

Proof. The solution of problem (1)–(3) for $f = f_k$, $k = 1,2$, denote $u_k(x,t)$. Let's use equality (41) and set in it $h = h_k$, $f = f_k$, $u = u_k$, $u_t = v_k$, $u_{tt} = w_k$. In addition, we set $t = 2x$. Then we get two equalities

$$\begin{aligned} 2[h_k'(2x) + f_k''(2x)] &= \sigma(x) b_k^m + q(x) a_k^p + \sigma(x)\big[(v^k(x,x))^m - b_k^m\big] \\ &+ 2\int_0^x \big[m\sigma(\xi) v_k^{m-1}(\xi, 2x-\xi) w_k(\xi, 2x-\xi) \\ &+ pq(\xi) u_k^{p-1}(\xi, 2x-\xi) v_k(\xi, 2x-\xi)\big]\,d\xi, \quad x \in [0, T/2], \quad k = 1,2. \end{aligned}$$

From these equalities we find that

$$\begin{aligned} \sigma(x) &= \sigma_0(x) + \frac{1}{D}\bigg\{\Big(a_2^p\big[v_1^m(x,x) - b_1^m\big] - a_1^p\big[v_2^m(x,x) - b_2^m\big]\Big)\sigma(x) \\ &+ 2a_2^p \int_0^x \big[m\sigma(\xi) v_1^{m-1}(\xi, 2x-\xi) w_1(\xi, 2x-\xi) + pq(\xi) u_1^{p-1}(\xi, 2x-\xi) v_1(\xi, 2x-\xi)\big]\,d\xi \end{aligned}$$

$$- 2a_1^p \int\limits_0^x \big[m\sigma(\xi)v_2^{m-1}(\xi, 2x-\xi)w_2(\xi, 2x-\xi)$$

$$+ pq(\xi)u_2^{p-1}(\xi, 2x-\xi)v_2(\xi, 2x-\xi)\big]\, d\xi \Bigg\}, \quad (48)$$

$$q(x) = q_0(x) + \frac{1}{D}\Bigg\{\Big(-b_2^m\big[v_1^m(x,x) - b_1^m\big] + b_1^m\big[v_2^m(x,x) - b_2^m\big]\Big)\sigma(x)$$

$$+ 2b_2^m \int\limits_0^x [m\sigma(\xi)v_1^{m-1}(\xi, 2x-\xi)w_1(\xi, 2x-\xi) + pq(\xi)u_1^{p-1}(\xi, 2x-\xi)v_1(\xi, 2x-\xi)]\, d\xi$$

$$+ 2b_1^m \int\limits_0^x [m\sigma(\xi)v_2^{m-1}(\xi, 2x-\xi)w_2(\xi, 2x-\xi)$$

$$+ pq(\xi)u_2^{p-1}(\xi, 2x-\xi)v_2(\xi, 2x-\xi)]\, d\xi \Bigg\}, \quad (49)$$

where

$$\begin{aligned} \sigma_0(x) &= \frac{1}{D}\big\{2a_1^p\big[h_2'(2x) + f_2''(2x)\big] - 2a_2^p\big[h_1'(2x) + f_1''(2x)\big]\big\}, \\ q_0(x) &= -\frac{1}{D}\big\{2b_1^m\big[h_2'(2x) + f_2''(2x)\big] - 2b_2^m\big[h_1'(2x) + f_1''(2x)\big]\big\}. \end{aligned} \quad (50)$$

Let's write out alternative equations for the functions $u_k(x,t)$, $v_k(x,t)$ and $w_k(x,t)$ for $k = 1, 2$. To do this, we apply the D'Alembert formula to problem (1), (3), (4), where we set $f = f_k$, $h = h_k$, $k = 1, 2$,

$$\begin{gathered} (u_k)_{tt} - (u_k)_{xx} + \sigma(x)(u_k)_t^m + q(x)u_k^p = 0, \quad x > 0, \quad t \in (0, T], \\ u_k|_{x=0} = f_k(t), \quad (u_k)_x|_{x=0} = h_k(t), \quad t \in [0, T]. \end{gathered} \quad (51)$$

As a result, we get the equation

$$u_k(x,t) = u_{0k}(x,t) - \frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{x+t-\xi} \big[\sigma(\xi)v_k^m(\xi,\tau) + q(\xi)u_k^p(\xi,\tau)\big]\, d\tau d\xi, \quad (52)$$

in which

$$u_{0k}(x,t) = \frac{f_k(t+x) + f_k(t-x)}{2} + \frac{1}{2}\int\limits_{t-x}^{t+x} h_k(\tau)\, d\tau. \quad (53)$$

Differentiate (52) with respect to t. Then we get the relations

$$v_k(x,t) = v_{0k}(x,t) - \frac{1}{2}\int\limits_0^x \Big[\big[\sigma(\xi) v_k^m(\xi, t+x-\xi) + q(\xi) u_k^p(\xi, t+x-\xi)\big] - \big[\sigma(\xi) v_k^m(\xi, t-x+\xi) + q(\xi) u_k^p(\xi, t-x+\xi)\big]\Big]\, d\xi, \quad (54)$$

$$\begin{aligned} w_k(x,t) = w_{0k}(x,t) - \frac{1}{2}\int\limits_0^x \Big[&\big[m\sigma(\xi) v_k^{m-1}(\xi, t+x-\xi) w_k(\xi, t+x-\xi) \\ &+ pq(\xi) u_k^{p-1}(\xi, t+x-\xi) v_k(\xi, t+x-\xi)\big] \\ &- \big[m\sigma(\xi) v_k^{m-1}(\xi, t-x+\xi) w_k(\xi, t-x+\xi) \\ &+ pq(\xi) u_k^{p-1}(\xi, t-x+\xi) v_k(\xi, t-x+\xi)\big]\Big]\, d\xi, \end{aligned} \quad (55)$$

where

$$\begin{aligned} v_{0k}(x,t) &= \frac{\partial u_{0k}}{\partial t}(x,t) = \frac{f_k'(t+x) + f_k'(t-x)}{2} + \frac{h_k(t+x) - h_k(t-x)}{2}, \\ w_{0k}(x,t) &= \frac{\partial^2 u_{0k}}{\partial t^2}(x,t) = \frac{f_k''(t+x) + f_k''(t-x)}{2} + \frac{h_k'(t+x) - h_k'(t-x)}{2}. \end{aligned} \quad (56)$$

Conditions (45) ensure that

$$u_{0k}(x,t) \geqslant \mu/2, \quad v_{0k}(x,t) \geqslant \mu/2, \quad (x,t) \in G(T). \quad (57)$$

Therefore, the solutions of equations (52), (54) will remain positive in the region $G(T)$, at least for sufficiently small values of T, which ones, we will find below.

Let's write down equations (48), (49), (52), (54), (55) in the operator form

$$\mathbf{g} = \widehat{\mathbf{A}}\mathbf{g}, \quad (58)$$

where

$$\begin{aligned} \mathbf{g} &= \big(\sigma(x), q(x), u_1(x,t), u_2(x,t), v_1(x,t), v_2(x,t), w_1(x,t), w_2(x,t)\big), \\ \mathbf{g}^0(x,t) &= \big(\sigma_0(x), q_0(x), u_{01}(x,t), u_{02}(x,t), v_{01}(x,t), v_{02}(x,t), w_{01}(x,t), w_{02}(x,t)\big). \end{aligned}$$

Functions $\sigma_0(x)$, $q_0(x)$, $u_{01}(x,t)$, $u_{02}(x,t)$, $v_{01}(x,t)$, $v_{02}(x,t)$, $w_{01}(x,t)$, $w_{02}(x,t)$ are defined by the equalities (50), (53), (56). Operators $\widehat{A}_1(\mathbf{g})$, $\widehat{A}_1(\mathbf{g})$, $\widehat{A}_{3+k}(\mathbf{g})$, $\widehat{A}_{4+k}(\mathbf{g})$, $\widehat{A}_{5+k}(\mathbf{g})$, $k = \overline{1,2}$, they are defined as follows:

$$\begin{aligned} \widehat{A}_1(\mathbf{g}) = \sigma_0(x) + \frac{1}{D}\Bigg\{ &\Big(a_2^p\big[v_1^m(x,x) - b_1^m\big] - a_1^p\big[v_2^m(x,x) - b_2^m\big]\Big)\sigma(x) \\ &+ 2a_2^p \int\limits_0^x \Big[m\sigma(\xi) v_1^{m-1}(\xi, 2x-\xi) w_1(\xi, 2x-\xi) + pq(\xi) u_1^{p-1}(\xi, 2x-\xi) v_1(\xi, 2x-\xi)\Big]\, d\xi \\ &- 2a_1^p \int\limits_0^x \Big[m\sigma(\xi) v_2^{m-1}(\xi, 2x-\xi) w_2(\xi, 2x-\xi) \end{aligned}$$

$$+ pq(\xi)u_2^{p-1}(\xi, 2x-\xi)v_2(\xi, 2x-\xi)\Big]\,d\xi\Big\}, \quad (59)$$

$$\widehat{A}_2(\mathbf{g}) = q_0(x) + \frac{1}{D}\Big\{\Big(-b_2^m\big[v_1^m(x,x) - b_1^m\big] + b_1^m\big[v_2^m(x,x) - b_2^m\big]D\Big)\sigma(x)$$
$$- 2b_2^m\int\limits_0^x \Big[m\sigma(\xi)v_1^{m-1}(\xi, 2x-\xi)w_1(\xi, 2x-\xi) + pq(\xi)u_1^{p-1}(\xi, 2x-\xi)v_1(\xi, 2x-\xi)\Big]\,d\xi$$
$$+ 2b_1^m\int\limits_0^x \Big[m\sigma(\xi)v_2^{m-1}(\xi, 2x-\xi)w_2(\xi, 2x-\xi)$$
$$+ pq(\xi)u_2^{p-1}(\xi, 2x-\xi)v_2(\xi, 2x-\xi)\Big]\,d\xi\Big\}, \quad (60)$$

$$\widehat{A}_{2+k}(\mathbf{g}) = u_{0k}(x,t) - \frac{1}{2}\int\limits_0^x\int\limits_{t-x+\xi}^{x+t-\xi}\big[\sigma(\xi)v_k^m(\xi,\tau) + q(\xi)u_k^p(\xi,\tau)\big]\,d\tau d\xi, \quad k=1,2, \quad (61)$$

$$\widehat{A}_{4+k}(\mathbf{g}) = v_{0k}(x,t) - \frac{1}{2}\int\limits_0^x \Big[\big[\sigma(\xi)v_k^m(\xi, t+x-\xi) + q(\xi)u_k^p(\xi, t+x-\xi)\big]$$
$$- \big[\sigma(\xi)v_k^m(\xi, t-x+\xi) + q(\xi)u_k^p(\xi, t-x+\xi)\big]\Big]\,d\xi, \quad k=1,2, \quad (62)$$

$$\widehat{A}_{6+k}(\mathbf{g}) = w_{0k}(x,t) - \frac{1}{2}\int\limits_0^x \Big[\big[m\sigma(\xi)v_k^{m-1}(\xi, t+x-\xi)w_k(\xi, t+x-\xi)$$
$$+ pq(\xi)u_k^{p-1}(\xi, t+x-\xi)v_k(\xi, t+x-\xi)\big] - \big[m\sigma(\xi)v_k^{m-1}(\xi, t-x+\xi)w_k(\xi, t-x+\xi)$$
$$+ pq(\xi)u_k^{p-1}(\xi, t-x+\xi)v_k(\xi, t-x+\xi)\big]\Big]\,d\xi, \quad k=1,2. \quad (63)$$

Operators $\widehat{A}_{2+k}$ and $\widehat{A}_{4+k}$ are defined on functions $(u_k(x,t), v_k(x,t))$, $k=1,2$, operators $\widehat{A}_{6+k}$ are defined on functions $(u_k(x,t), v_k(x,t), w_k(x,t))$, $k=1,2$.

Let's denote $\mathbf{C}(G(T))$ the space of continuous vector functions with norm

$$\|\mathbf{g}\|_{\mathbf{C}(G(T))} = \max\big\{\|\sigma\|_{C[0,T/2]},\ \|q\|_{C[0,T/2]},\ \|u_k\|_{C(G(T))},\ \|v_k\|_{C(G(T))},\ \|w_k\|_{C(G(T))}\big\}.$$

Since $\mathbf{g}^0 \in \mathbf{C}(G(T))$, then all vector functions defined in (59)–(63) are elements of $\mathbf{C}(G(T))$, and from (50), (53), (56) it follows

$$\|\mathbf{g}^0\|_{\mathbf{C}(G(T))} = \max\big\{\|\sigma_0\|_{C[0,T/2]},\ \|q_0\|_{C[0,T/2]},\ \|u_{0k}\|_{C(G(T))},\ \|v_{0k}\|_{C(G(T))},\ \|w_{0k}\|_{C(G(T))}\big\}$$
$$\leqslant \max\{4\beta^p(F+H)/D,\ F+HT/2,\ F+H\} =: M. \quad (64)$$

Here

$$\beta := \max\{a_1, b_1, a_2, b_2\}. \quad (65)$$

Consider in the Banach space $\mathbf{C}(G(T))$ the closed set

$$\mathcal{M}(T,M,\mu) := \big\{\mathbf{g} \in \mathbf{C}(G(T)) \mid \big\|\sigma - \sigma_0\big\|_{C[0,T/2]} \leqslant M,\ \big\|q - q_0\big\|_{C[0,T/2]} \leqslant M,$$
$$\big\|u_k - u_{0k}\big\|_{C(G(T))} \leqslant M,\ \big\|v_k - v_{0k}\big\|_{C(G(T))} \leqslant M,\ \big\|w_k - w_{0k}\big\|_{C(G(T))} \leqslant M\big\},$$
$$u_k(x,t) \geqslant \mu/4, \quad v_k(x,t) \geqslant \mu/4, \quad k = 1,2. \quad (66)$$

Following estimates hold on this set

$$\|\sigma\|_{C[0,T/2]} \leqslant 2M, \quad \|q\|_{C[0,T/2]} \leqslant 2M,$$
$$\|u_k\|_{C(G(T))} \leqslant 2M, \quad \|v_k\|_{C(G(T))} \leqslant 2M, \quad \|w_k\|_{C(G(T))} \leqslant 2M, \quad k = 1,2. \quad (67)$$

From equalities (59)–(63) we have

$$\widehat{A}_1(\mathbf{g}) - \sigma_0(x) = \frac{1}{D}\Bigg\{\Big(a_2^p\big[v_1^m(x,x) - b_1^m\big] - a_1^p\big[v_2^m(x,x) - b_2^m\big]\Big)\sigma(x)$$
$$+ 2a_2^p\int_0^x \big[m\sigma(\xi)v_1^{m-1}(\xi,2x-\xi)w_1(\xi,2x-\xi) + pq(\xi)u_1^{p-1}(\xi,2x-\xi)v_1(\xi,2x-\xi)\big]\,d\xi$$
$$- 2a_1^p\int_0^x \big[m\sigma(\xi)v_2^{m-1}(\xi,2x-\xi)w_2(\xi,2x-\xi)$$
$$+ pq(\xi)u_2^{p-1}(\xi,2x-\xi)v_2(\xi,2x-\xi)\big]\,d\xi\Bigg\}, \quad (68)$$

$$\widehat{A}_2(\mathbf{g}) - q_0(x) = \frac{1}{D}\Bigg\{\Big(-b_2^m\big[v_1^m(x,x) - b_1^m\big] + b_1^m\big[v_2^m(x,x) - b_2^m\big]\Big)\sigma(x)$$
$$2b_2^m\int_0^x \big[m\sigma(\xi)v_1^{m-1}(\xi,2x-\xi)w_1(\xi,2x-\xi) + pq(\xi)u_1^{p-1}(\xi,2x-\xi)v_1(\xi,2x-\xi)\big]\,d\xi$$
$$+ 2b_1^m\int_0^x \big[m\sigma(\xi)v_2^{m-1}(\xi,2x-\xi)w_2(\xi,2x-\xi)$$
$$+ pq(\xi)u_2^{p-1}(\xi,2x-\xi)v_2(\xi,2x-\xi)\big]\,d\xi\Bigg\}, \quad (69)$$

$$\widehat{A}_{2+k}(\mathbf{g}) - u_{0k}(x,t) = -\frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{x+t-\xi} \big[\sigma(\xi)v_k^m(\xi,\tau) + q(\xi)u_k^p(\xi,\tau)\big]\,d\tau d\xi, \quad k=1,2, \tag{70}$$

$$\widehat{A}_{4+k}(\mathbf{g}) - v_{0k}(x,t) = -\frac{1}{2}\int\limits_0^x \Big[\big[\sigma(\xi)v_k^m(\xi,t+x-\xi) + q(\xi)u_k^p(\xi,t+x-\xi)\big] \\ - \big[\sigma(\xi)v_k^m(\xi,t-x+\xi) + q(\xi)u_k^p(\xi,t-x+\xi)\big]\Big]\,d\xi, \quad k=1,2, \tag{71}$$

$$\widehat{A}_{6+k}(\mathbf{g}) - w_{0k}(x,t) = -\frac{1}{2}\int\limits_0^x \Big[\big[m\sigma(\xi)v_k^{m-1}(\xi,t+x-\xi)w_k(\xi,t+x-\xi) \\ + pq(\xi)u_k^{p-1}(\xi,t+x-\xi)v_k(\xi,t+x-\xi)\big] - \big[m\sigma(\xi)v_k^{m-1}(\xi,t-x+\xi)w_1(\xi,t-x+\xi) \\ + pq(\xi)u_k^{p-1}(\xi,t-x+\xi)v_k(\xi,t-x+\xi)\big]\Big]\,d\xi, \quad k=1,2. \tag{72}$$

Let $\varphi_k \in C(G(T))$, $k=1,2$. Consider the difference of the functions $\varphi_1^m(x,t) - \varphi_2^m(x,t)$, $m>1$, and represent it as follows:

$$\varphi_1^m(x,t) - \varphi_2^m(x,t) = m\int\limits_{\varphi_1(x,t)}^{\varphi_2(x,t)} s^{m-1}\,ds = \big(\varphi_1(x,t) - \varphi_2(x,t)\big)R_m[\varphi_1,\varphi_2](x,t), \tag{73}$$

where

$$R_m[\varphi_1,\varphi_2](x,t) = m\int\limits_0^1 \big[\varphi_1(x,t)s' + \varphi_2(x,t)(1-s')\big]^{m-1}\,ds' \\ \leqslant m \max_{(x,t)\in G(T)}\{\varphi_1(x,t),\varphi_2(x,t)\}. \tag{74}$$

Using (73) (74), the difference $v_k^m(x,x) - b_k^m$ can be written as follows:

$$v_k^m(x,x) - b_k^m = (v_k(x,x) - b_k)R_m[v_k,b_k](x), \\ R_m[v_k,b_k](x) = m\int\limits_0^1 \big[v_k(x,x)s' + b_k(1-s')\big]^{m-1}\,ds', \\ |R_m[v_k,b_k](x)| \leqslant m(2M)^{m-1} =: R_m^*. \tag{75}$$

Let's put $t=x$ in formula (8), Then we get the equality

$$v_k(x,x) = f'(0) + \frac{1}{2}\int\limits_0^x \big[\sigma(\xi)v_k^m(\xi,\xi) + q(\xi)u_k^p(\xi,\xi)\big]\,d\xi, \quad k=1,2. \tag{76}$$

From here we find

$$|v_k(x,x) - b_k| \leqslant \frac{1}{2}\int\limits_0^x \big|\sigma(\xi)v_k^m(\xi,\xi) + q(\xi)u^p(\xi,\xi)\big|\, d\xi$$

$$\leqslant \big[(2M)^{m+1} + (2M)^{p+1}\big]T/4 =: C_0T, \quad k = 1,2. \quad (77)$$

Considering (75), (77), from equations (68)–(72) we find estimates

$$|\widehat{A}_1(\mathbf{g}) - \sigma_0(x)| \leqslant \frac{1}{D}\bigg\{\Big(a_2^p\big|v_1^m(x,x) - b_1^m\big| + a_1^p\big|v_2^m(x,x) - b_2^m\big|\Big)|\sigma(x)|$$

$$+ 2a_2^p\int\limits_0^x \big|m\sigma(\xi)v_1^{m-1}(\xi,2x-\xi)w_1(\xi,2x-\xi) + pq(\xi)u_1^{p-1}(\xi,2x-\xi)v_1(\xi,2x-\xi)\big|\, d\xi$$

$$+ 2a_1^p\int\limits_0^x \big|m\sigma(\xi)v_2^{m-1}(\xi,2x-\xi)w_2(\xi,2x-\xi) + pq(\xi)u_2^{p-1}(\xi,2x-\xi)v_2(\xi,2x-\xi)\big|\, d\xi$$

$$\leqslant \frac{2\beta^p}{D}\big(R_p^*C_0 + 2m(2M)^{m+1} + 2p(2M)^{p+1}\big)T =: C_1T,$$

$$|\widehat{A}_2(\mathbf{g}) - q_0(x)| \leqslant \frac{1}{D}\bigg\{\Big(b_2^m\big|v_1^m(x,x) - b_1^m\big| + b_1^m\big|v_2^m(x,x) - b_2^m\big|\Big)|\sigma(x)|$$

$$+ 2b_2^m\int\limits_0^x \big|m\sigma(\xi)v_1^{m-1}(\xi,2x-\xi)w_1(\xi,2x-\xi) + pq(\xi)u_1^{p-1}(\xi,2x-\xi)v_1(\xi,2x-\xi)\big|\, d\xi$$

$$+ 2b_1^m\int\limits_0^x \big|m\sigma(\xi)v_2^{m-1}(\xi,2x-\xi)w_2(\xi,2x-\xi) + pq(\xi)u_2^{p-1}(\xi,2x-\xi)v_2(\xi,2x-\xi)\big|\, d\xi\bigg\}$$

$$\leqslant \frac{2\beta^m}{D}\big(R_m^*C_0 + 2m(2M)^{m+1} + 2p(2M)^{p+1}\big)T =: C_2T,$$

$$|\widehat{A}_{2+k}(\mathbf{g}) - u_{0k}(x,t)| \leqslant \frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{x+t-\xi} \big|\sigma(\xi)v_k^m(\xi,\tau) + q(\xi)u_k^p(\xi,\tau)\big|\, d\tau d\xi$$

$$\leqslant [(2M)^{m+1} + (2M)^{p+1}]\frac{T^2}{4} =: C_3T^2, \quad k = 1,2,$$

$$|\widehat{A}_{4+k}(\mathbf{g}) - v_{0k}(x,t)| \leqslant \frac{1}{2}\int\limits_0^x \Big[\big|\sigma(\xi)v_k^m(\xi,t+x-\xi) + q(\xi)u_k^p(\xi,t+x-\xi)\big|$$

$$+ \big|\sigma(\xi)v_k^m(\xi,t-x+\xi) + q(\xi)u_k^p(\xi,t-x+\xi)\big|\Big]\, d\xi$$

$$\leqslant \big[(2M)^{m+1}+(2M)^{p+1}\big]\frac{T}{2}=: C_4T,\quad k=1,2,$$

$$|\widehat{A}_{6+k}(\mathbf{g})-w_{0k}(x,t)|\leqslant\frac{1}{2}\int\limits_0^x\Big[\big|m\sigma(\xi)v_k^{m-1}(\xi,t+x-\xi)w_k(\xi,t+x-\xi)$$
$$+pq(\xi)u_k^{p-1}(\xi,t+x-\xi)v_k(\xi,t+x-\xi)\big|+\big|m\sigma(\xi)v_k^{m-1}(\xi,t-x+\xi)w_k(\xi,t-x+\xi)$$
$$+pq(\xi)u_k^{p-1}(\xi,t-x+\xi)v_k(\xi,t-x+\xi)\big|\Big]\,d\xi$$
$$\leqslant\big[m(2M)^{m+1}+p(2M)^{p+1}\big]\frac{T}{2}=: C_5T,\quad k=1,2.$$

Using the inequality (57), we obtain

$$u_k(x,t)=u_{0k}(x,t)+[u_k(x,t)-u_{0k}(x,t)]\geqslant\mu/2-C_3T^2\geqslant\mu/4,$$
$$v_k(x,t)=v_{0k}(x,t)+[v_k(x,t)-v_{0k}(x,t)]\geqslant\mu/2-C_4T\geqslant\mu/4,\quad k=1,2,$$

for

$$T\leqslant\min\big\{\sqrt{\mu/(4C_3)},\mu/(4C_4)\big\}.$$

Let's select T_0' from the condition

$$T_0'=\min\big\{T,\sqrt{\mu/(4C_3)},\mu/(4C_4),M/C_1,M/C_2,M/C_5\big\}.$$

Then the inequalities $u_k(x,t)\geqslant\mu/4$, $v_k(x,t)\geqslant\mu/4$, $(x,t)\in G(T_0')$, $k=1,2$, and inequalities (66) are satisfied. It follows from here that the operator $\widehat{\mathbf{A}}(\mathbf{g})$ maps the set $\mathcal{M}(T_0',M,\mu)$ into yourself. Below we will assume that $T\leqslant T_0'$.

Let us now demonstrate that the operator $\widehat{\mathbf{A}}$, defined by equalities (59)–(63), is compressive for a sufficiently small $T\leqslant T_0'$.

Let $\mathbf{g}^k\in\mathbf{C}(G(T))$, $k=1,2$,

$$\mathbf{g}^1=\big(\sigma^1(x),q^1(x),u_1^1(x,t),u_2^1(x,t),v_1^1(x,t),v_2^1(x,t),w_1^1(x,t),w_2^1(x,t)\big),$$
$$\mathbf{g}^2=\big(\sigma^2(x),q^2(x),u_1^2(x,t),u_2^2(x,t),v_1^2(x,t),v_2^2(x,t),w_1^2(x,t),w_2^2(x,t)\big),$$

$$\overline{\sigma}(x)=\sigma^1(x)-\sigma^2(x),\quad \overline{q}(x)=q^1(x)-q^2(x),$$
$$\overline{u}_k(x,t)=u_k^1(x,t)-u_k^2(x,t),\quad \overline{v}_k(x,t)=v_k^1(x,t)-v_k^2(x,t),$$
$$\overline{w}_k(x,t)=w_k^1(x,t)-w_k^2(x,t),\quad k=1,2.$$

From (73)–(75) the estimates follow

$$\begin{gathered}\big|v_1^m(x,t)-v_2^m(x,t)\big|\leqslant|R_m[v_1,v_2](x,t)||\overline{v}(x,t)|,\\ \big|u_1^p(x,t)-u_2^p(x,t)\big|\leqslant|R_p[u_1,u_2](x,t)||\overline{u}(x,t)|,\\ R_m^*=m(2M)^{m-1},\quad R_p^*=p(2M)^{p-1}.\end{gathered}\tag{78}$$

Let's use equality (59) and write down the difference

$$
\begin{aligned}
&\widehat{A}_1(\mathbf{g}^1) - \widehat{A}_1(\mathbf{g}^2)\\
&= \frac{1}{D}\Bigg\{ \Big(a_2^p \big[\big(v_1^1(x,x)\big)^m - \big(v_1^2(x,x)\big)^m\big] - a_1^p \big[\big(v_2^1(x,x)\big)^m - \big(v_2^2(x,x)\big)^m\big]\Big)\sigma_1(x)\\
&\quad + \Big(a_2^p \big[(v_1^2(x,x))^m - b_1^m\big] - a_1^p \big[(v_2^2(x,x))^m - b_2^m\big]\Big)\overline{\sigma}(x)\\
&\quad + 2a_2^p \int_0^x \Big[m\overline{\sigma}(\xi)\big(v_1^1(\xi,2x-\xi)\big)^{m-1} w_1^1(\xi,2x-\xi)\\
&\quad + m\sigma^2(\xi)\big[\big(v_1^1(\xi,2x-\xi)\big)^{m-1} - \big(v_1^2(\xi,2x-\xi)\big)^{m-1}\big] w_1^1(\xi,2x-\xi)\\
&\quad + m\sigma^2(\xi)\big(v_1^2(\xi,2x-\xi)\big)^{m-1}\overline{w}_1(\xi,2x-\xi) + p\overline{q}(\xi)\big(u_1^1(\xi,2x-\xi)\big)^{p-1} v_1^1(\xi,2x-\xi)\\
&\quad + pq^2(\xi)\big[\big(u_1^1(\xi,2x-\xi)\big)^{p-1} - \big(u_1^2(\xi,2x-\xi)\big)^{p-1}\big] v_1^1(\xi,2x-\xi)\\
&\quad + pq^2(\xi)\big(u_1^2(\xi,2x-\xi)\big)^{p-1}\overline{v}_1(\xi,2x-\xi)\Big]\\
&\quad - 2a_1^p \int_0^x \Big[m\overline{\sigma}(\xi)\big(v_2^1(\xi,2x-\xi)\big)^{m-1} w_2^1(\xi,2x-\xi)\\
&\quad + m\sigma^2(\xi)\big[\big(v_2^1(\xi,2x-\xi)\big)^{m-1} - \big(v_2^2(\xi,2x-\xi)\big)^{m-1}\big] w_2^1(\xi,2x-\xi)\\
&\quad + m\sigma^2(\xi)\big(v_2^2(\xi,2x-\xi)\big)^{m-1}\overline{w}_2(\xi,2x-\xi) + p\overline{q}(\xi)\big(u_2^1(\xi,2x-\xi)\big)^{p-1} v_2^1(\xi,2x-\xi)\\
&\quad + pq^2(\xi)\big[\big(u_2^1(\xi,2x-\xi)\big)^{p-1} - \big(u_2^2(\xi,2x-\xi)\big)^{p-1}\big] v_2^1(\xi,2x-\xi)\\
&\quad + pq^2(\xi)\big(u_2^2(\xi,2x-\xi)\big)^{p-1}\overline{v}_2(\xi,2x-\xi)\Big]\Bigg\}, \qquad (79)
\end{aligned}
$$

Similarly, from formula (60) we find

$$
\begin{aligned}
&\widehat{A}_2(\mathbf{g}^1) - \widehat{A}_2(\mathbf{g}^2)\\
&= \frac{1}{D}\Bigg\{ \Big(-b_2^m \big[\big(v_1^1(x,x)\big)^m - \big(v_1^2(x,x)\big)^m\big] + b_1^m \big[\big(v_1^1(x,x)\big)^m - \big(v_2^2(x,x)\big)^m\big]\Big)\sigma_1(x)\\
&\quad + \Big(-b_2^m \big[\big(v_1^2(x,x)\big)^m - b_1^m\big] + b_1^m \big[\big(v_2^1(x,x)\big)^m - b_2^m\big]\Big)\overline{\sigma}(x)\\
&\quad - 2b_2^m \int_0^x \Big[m\overline{\sigma}(\xi)\big(v_1^1(\xi,2x-\xi)\big)^{m-1} w_1^1(\xi,2x-\xi)\\
&\quad + m\sigma^2(\xi)\big[\big(v_1^1(\xi,2x-\xi)\big)^{m-1} - \big(v_1^2(\xi,2x-\xi)\big)^{m-1}\big] w_1^1(\xi,2x-\xi)\\
&\quad + m\sigma^2(\xi)\big(v_1^2(\xi,2x-\xi)\big)^{m-1}\overline{w}_1(\xi,2x-\xi) + p\overline{q}(\xi)\big(u_1^1(\xi,2x-\xi)\big)^{p-1} v_1^1(\xi,2x-\xi)\Big]\\
&\quad + pq^2(\xi)\big[\big(u_1^1(\xi,2x-\xi)\big)^{p-1} - \big(u_1^2(\xi,2x-\xi)\big)^{p-1}\big] v_1^1(\xi,2x-\xi)\\
&\quad + pq^2(\xi)\big(u_1^2(\xi,2x-\xi)\big)^{p-1}\overline{v}_1(\xi,2x-\xi)\big]\Big]\, d\xi\\
&\quad + 2b_1^m \int_0^x \Big[m\overline{\sigma}(\xi)\big(v_2^1(\xi,2x-\xi)\big)^{m-1} w_2^1(\xi,2x-\xi)
\end{aligned}
$$

$$+ m\sigma^2(\xi)\big[\big(v_2^1(\xi,2x-\xi)\big)^{m-1} - \big(v_2^2(\xi,2x-\xi)v\big)^{m-1}\big]w_2^1(\xi,2x-\xi)$$
$$+ m\sigma^2(\xi)\big(v_2^2(\xi,2x-\xi)\big)^{m-1}\overline{w}_2(\xi,2x-\xi) + p\overline{q}(\xi)\big(u_2^1(\xi,2x-\xi)\big)^{p-1}v_2^1(\xi,2x-\xi)$$
$$+ pq^2(\xi)\big[\big(u_2^1(\xi,2x-\xi)\big)^{p-1} - \big(u_2^2(\xi,2x-\xi)\big)^{p-1}\big]v_2^1(\xi,2x-\xi)$$
$$+ pq^2(\xi)\big(u_2^2(\xi,2x-\xi)\big)^{p-1}\overline{v}_2(\xi,2x-\xi)\Big]\,d\xi\Bigg\}, \quad (80)$$

Let's use the equations (67), (76), (78) and let's estimate the difference

$$|v_k^1(x,x) - v_k^2(x,x)| \leqslant \frac{1}{2}\int\limits_0^x \Big|\overline{\sigma}(\xi)(v_k^1(\xi,\xi))^m + \sigma^2(\xi)\big[\big(v_k^1(\xi,\xi)\big)^m - \big(v_k^2(\xi,\xi)\big)^m\big]$$
$$+ \overline{q}(\xi)u_k^p(\xi,\xi) + q^2(\xi)\big[\big(u_1(\xi,\xi)\big)^p - \big(u_1(\xi,\xi)\big)^p\big]\Big|\,d\xi$$
$$\leqslant \frac{T}{4}\big[(2M)^m + (2M)^2R_m^* + (2M)^p + (2M)^2R_p^*\big]\|\mathbf{g}^1 - \mathbf{g}^2\|$$
$$=: \alpha_0 T\|\mathbf{g}^1 - \mathbf{g}^2\|, \quad k=1,2, \quad (81)$$

From the formulas (78), (79), (81) it follows that

$$\big|\widehat{A}_1(\mathbf{g}^1) - \widehat{A}_1(\mathbf{g}^2)\big| \leqslant \frac{2\beta^p}{D}\Big((\alpha_0 + C_0)R_m^* + m\big[4(2M)^m + (2M)^2R_m^*\big]$$
$$+ p\big[4(2M)^p + (2M)^2R_p^*\big]\Big)T\|\mathbf{g}^1 - \mathbf{g}^2\|$$
$$=: \alpha_1 T\|\mathbf{g}^1 - \mathbf{g}^2\|. \quad (82)$$

A similar estimate is true for the difference $|\widehat{A}_2(\mathbf{g}^1) - \widehat{A}_2(\mathbf{g}^2)|$:

$$\big|\widehat{A}_2(\mathbf{g}^1) - \widehat{A}_2(\mathbf{g}^2)\big| \leqslant \frac{2\beta^m}{D}\Big((\alpha_0 + C_0)R_m^* + m\big[4(2M)^m + (2M)^2R_m^*\big]$$
$$+ p\big[4(2M)^p + (2M)^2R_p^*\big]\Big)T\|\mathbf{g}^1 - \mathbf{g}^2\|$$
$$= \alpha_2 T\|\mathbf{g}^1 - \mathbf{g}^2\|. \quad (83)$$

Let's use equality (61) and write down the difference

$$\widehat{A}_{2+k}(\mathbf{g}^1) - \widehat{A}_{2+k}(\mathbf{g}^2) = -\frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{x+t-\xi} \Big[\overline{\sigma}(\xi)(v_k^1(\xi,\tau))^m + \sigma^2(\xi)\big[\big(v_k^1(\xi,\tau)\big)^m - \big(v_k^2(\xi,\tau)\big)^m\big]$$
$$+ \overline{q}(\xi)\big(u_k^1(\xi,\tau)\big)^p + q^2(\xi)\big[\big(u_k^1(\xi,\tau)\big)^p - \big(u_k^2(\xi,\tau)\big)^p\big]\Big]\,d\tau d\xi, \quad k=1,2. \quad (84)$$

Estimate this difference as

$$|\widehat{A}_{2+k}(\mathbf{g}^1) - \widehat{A}_{2+k}(\mathbf{g}^2)| \leqslant \frac{1}{2}\int\limits_0^x \int\limits_{t-x+\xi}^{x+t-\xi} \Big|\overline{\sigma}(\xi)(v_k^1(\xi,\tau))^m + \sigma^2(\xi)\big[(v_k^1(\xi,\tau))^m - (v_k^2(\xi,\tau))^m\big]$$

$$+ \overline{q}(\xi)(u_k^1(\xi,\tau))^p + q^2(\xi)\big[(u_k^1(\xi,\tau))^p - (u_k^2(\xi,\tau))^p\big]\Big| d\tau d\xi$$
$$\leqslant \frac{T^2}{4}\big[(2M)^m + (2M)R_m^* + (2M)^p + (2M)R_p^*\big]\|\mathbf{g}^1 - \mathbf{g}^2\|$$
$$=: \alpha_3 T^2\|\mathbf{g}^1 - \mathbf{g}^2\|, \quad k = 1,2. \quad (85)$$

From equation (62) we find

$$\widehat{A}_{4+k}(\mathbf{g}^1) - \widehat{A}_{4+k}(\mathbf{g}^2) = -\frac{1}{2}\int_0^x \Big[\overline{\sigma}(\xi)(v_k^1(\xi,t+x-\xi))^m$$
$$+ \sigma^2(\xi)\big[(v_k^1(\xi,t+x-\xi))^m - (v_k^2(\xi,t+x-\xi))^m\big]$$
$$+ \overline{q}(\xi)(u_k^1(\xi,t-x+\xi))^p + q^2(\xi)\big[(u_k^1(\xi,t-x+\xi))^p - (u_k^2(\xi,t-x+\xi))^p\big]$$
$$- \Big(\overline{\sigma}(\xi)(v_k^1(\xi,t-x+\xi))^m + \sigma^2(\xi)\big[(v_k^1(\xi,t-x+\xi))^m - (v_k^2(\xi,t-x+\xi))^m\big]$$
$$+ \overline{q}(\xi)(u_k^1(\xi,t-x+\xi))^p + q^2(\xi)\big[(u_k^1(\xi,t-x+\xi))^p - [(u_k^2(\xi,t-x+\xi))^p\big]\Big)\Big] d\xi,$$
$$k = 1,2. \quad (86)$$

Estimate this difference as follows.

$$|\widehat{A}_{4+k}(\mathbf{g}^1) - \widehat{A}_{4+k}(\mathbf{g}^2)| \leqslant \frac{1}{2}\int_0^x \Big|\overline{\sigma}(\xi)(v_k^1(\xi,t+x-\xi))^m$$
$$+ \sigma^2(\xi)\big[\big(v_k^1(\xi,t+x-\xi)\big)^m - \big(v_k^2(\xi,t+x-\xi)\big)^m\big]$$
$$+ \overline{q}(\xi)\big(u_k^1(\xi,t-x+\xi)\big)^p + q^2(\xi)\big[\big(u_k^1(\xi,t-x+\xi)\big)^p - \big(u_k^2(\xi,t-x+\xi)\big)^p\big]$$
$$+ \overline{\sigma}(\xi)\big(v_k^1(\xi,t-x+\xi)\big)^m + \sigma^2(\xi)\big[\big(v_k^1(\xi,t-x+\xi)\big)^m - \big(v_k^2(\xi,t-x+\xi)\big)^m\big]$$
$$+ \overline{q}(\xi)\big(u_k^1(\xi,t-x+\xi)\big)^p + q^2(\xi)\big[\big(u_k^1(\xi,t-x+\xi)\big)^p - \big(u_k^2(\xi,t-x+\xi)\big)^p\big]\Big| d\xi$$
$$\leqslant \frac{T}{2}\big[2(2M)^m + (2M)R_m^* + 2(2M)^p + (2M)R_p^*\big]\|\mathbf{g}^1 - \mathbf{g}^2\|$$
$$=: \alpha_4 T\|\mathbf{g}^1 - \mathbf{g}^2\|, \quad k = 1,2. \quad (87)$$

Using equality (63), we obtain

$$\widehat{A}_{6+k}(\mathbf{g}^1) - \widehat{A}_{6+k}(\mathbf{g}^2) = -\frac{1}{2}\int_0^x \Big[m\overline{\sigma}(\xi)\big(v_k^1(\xi,t+x-\xi)\big)^{m-1}w_k^1(\xi,t+x-\xi)$$
$$+ m\sigma^2(\xi)\big[\big(v_k^1(\xi,t+x-\xi)\big)^{m-1} - \big(v_k^2(\xi,t+x-\xi)\big)^{m-1}\big]w_k^1(\xi,t+x-\xi)$$
$$+ m\sigma^2(\xi)\big(v_k^2(\xi,t+x-\xi)\big)^{m-1}\overline{w}_k(\xi,t+x-\xi) + p\overline{q}(\xi)\big(u_k^1(\xi,t+x-\xi)\big)^{p-1}v_k^1(\xi,t+x-\xi)$$
$$+ pq^2(\xi)\big[\big(u_k^1(\xi,t+x-\xi)\big)^{p-1} - \big(u_k^2(\xi,t+x-\xi)\big)^{p-1}\big]v_k^1(\xi,t+x-\xi)$$
$$+ pq^2(\xi)\big(u_k^2(\xi,t+x-\xi)\big)^{p-1}\overline{v}_k(\xi,t+x-\xi) + m\overline{\sigma}(\xi)\big(v_k^1(\xi,t-x+\xi)\big)^{m-1}w_1^1(\xi,t-x+\xi)$$
$$+ m\sigma^2(\xi)\big[\big(v_k^1(\xi,t-x+\xi)\big)^{m-1} - \big(v_k^2(\xi,t-x+\xi)\big)^{m-1}\big]w_1^1(\xi,t-x+\xi)$$

$$
\begin{aligned}
&+ m\sigma^2(\xi)\big(v_k^2(\xi,t-x+\xi)\big)^{m-1}\overline{w}_1(\xi,t-x+\xi)\\
&+ p\overline{q}(\xi)\big(u_k^1(\xi,t-x+\xi)\big)^{p-1}v_k^1(\xi,t-x+\xi)\\
&+ pq^2(\xi)\big[\big(u_k^1(\xi,t-x+\xi)\big)^{p-1}-\big(u_k^2(\xi,t-x+\xi)\big)^{p-1}\big]v_k^1(\xi,t-x+\xi)\\
&+ pq^2(\xi)\big(u_k^2(\xi,t-x+\xi)\big)^{p-1}\overline{v}_k(\xi,t-x+\xi)\Big]\,d\xi, \quad k=1,2. \qquad (88)
\end{aligned}
$$

Evaluating this difference, we find

$$
\begin{aligned}
|\widehat{A}_{6+k}(\mathbf{g}^1)-\widehat{A}_{6+k}(\mathbf{g}^2)| \leqslant \frac{1}{2}\int_0^x \Big| &m\overline{\sigma}(\xi)\big(v_k^1(\xi,t+x-\xi)\big)^{m-1}w_k^1(\xi,t+x-\xi)\\
&+ m\sigma^2(\xi)\big[\big(v_k^1(\xi,t+x-\xi)\big)^{m-1}-\big(v_k^2(\xi,t+x-\xi)\big)^{m-1}\big]w_k^1(\xi,t+x-\xi)\\
&+ m\sigma^2(\xi)\big(v_k^2(\xi,t+x-\xi)\big)^{m-1}\overline{w}_k(\xi,t+x-\xi)+p\overline{q}(\xi)\big(u_k^1(\xi,t+x-\xi)\big)^{p-1}v_k^1(\xi,t+x-\xi)\\
&+ pq^2(\xi)\big[\big(u_k^1(\xi,t+x-\xi)\big)^{p-1}-\big(u_k^2(\xi,t+x-\xi)\big)^{p-1}\big]v_k^1(\xi,t+x-\xi)\\
&+ pq^2(\xi)\big(u_k^2(\xi,t+x-\xi)\big)^{p-1}\overline{v}_k(\xi,t+x-\xi)+m\overline{\sigma}(\xi)\big(v_k^1(\xi,t-x+\xi)\big)^{m-1}w_1^1(\xi,t-x+\xi)\\
&+ m\sigma^2(\xi)\big[\big(v_k^1(\xi,t-x+\xi)\big)^{m-1}-\big(v_k^2(\xi,t-x+\xi)\big)^{m-1}\big]w_1^1(\xi,t-x+\xi)\\
&+ m\sigma^2(\xi)\big(v_k^2(\xi,t-x+\xi)\big)^{m-1}\overline{w}_1(\xi,t-x+\xi)\\
&+ p\overline{q}(\xi)\big(u_k^1(\xi,t-x+\xi)\big)^{p-1}v_k^1(\xi,t-x+\xi)\\
&+ pq^2(\xi)\big[\big(u_k^1(\xi,t-x+\xi)\big)^{p-1}-\big(u_k^2(\xi,t-x+\xi)\big)^{p-1}\big]v_k^1(\xi,t-x+\xi)\\
&+ pq^2(\xi)\big(u_k^2(\xi,t-x+\xi)\big)^{p-1}\overline{v}_k(\xi,t-x+\xi)\Big|\,d\xi\\
&\leqslant \frac{T}{2}\big[2m(2M)^m+m(2M)^2R^*_{m-1}+2p(2M)^p+p(2M)^2R^*_{p-1}\big]\|\mathbf{g}^1-\mathbf{g}^2\|\\
&=: \alpha_5 T\|\mathbf{g}^1-\mathbf{g}^2\|. \qquad (89)
\end{aligned}
$$

Take a number $\rho\in(0,1)$ and choose T_0 from the condition

$$T_0 := \min\big\{T_0',\ \rho/\alpha_1,\ \rho/\alpha_2,\ \sqrt{\rho/\alpha_3},\ \rho/\alpha_4,\ \rho/\alpha_5\big\}.$$

Then

$$\|\widehat{A}\mathbf{g}^1-\widehat{A}\mathbf{g}^2\|_{\mathbf{C}(G(T_0))}\leqslant \rho\|\mathbf{g}^1-\mathbf{g}^2\|_{\mathbf{C}(G(T_0))}.$$

Thus, the mapping $\widehat{A}$ is compressive on the set $\mathcal{M}(T_0,M,\mu)$. By virtue of the Banach principle of compressive maps we conclude that on set $\mathcal{M}(T_0,M,\mu)$ there exists a unique solution of operator equation (58). Theorem 4.2 is proved. □

Remark 4.2. The stability theorem can be obtained only for small T. The local stability theorem can be deduced from the Banach principle of compressive maps.

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Romanov V. G.,
Sobolev Institute of Mathematics,
Novosibirsk 630090, Koptyug prosp., 4, Russia,
Email: `romanov@math.nsc.ru`

Bugueva T. V.
Sobolev Institute of Mathematics,
Novosibirsk 630090, Koptyug prosp., 4, Russia,
Novosibirsk State University,
Novosibirsk 630090, ul. Pirogova 1, Russia,
Email: `bugueva@math.nsc.ru`

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