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STABILITY OF THE LINEARIZATION METHOD APPLIED IN TRICOMPARTMENTAL POLYNOMIAL CATENARY SYSTEMS OF ORDER $(\alpha + \beta)$

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Abstract This paper aims to identify exchange coefficients of a nonlinear polynomial tri-compartmental catenary system of $(\alpha + \beta)$ order. This is based on two principal procedures. The first procedure presented is related to the recommended solution consisting of introducing an adequate time $t^* > 0$ in a way to be defined. That is to say: wait a moment to allow the exchange to settle in the polynomial $(\alpha + \beta)$ order nonlinear catenary system after injecting the quantity into the main compartment, then measure this compartment with compartment 2, at this time $t^* > 0$. In the second procedure, we apply the Taylor formula to linearize the nonlinear system and identify the exchange coefficients. In the end, we will prove that the linearization method is stable.

Key words: Linear compartmental system, Nonlinear compartmental system, Inverse problem, Identification, Numerical analysis, Ordinary differential equation.

AMS Mathematics Subject Classification: 34A55, 93C15, 34A34.

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1 Introduction

In this paper, Nonlinear compartmental systems of polynomial type are encountered particularly in population dynamics. These systems are controlled by the following hypothesis: "The quantity passing from compartment i to compartment j is equal to $k_{ij}x_i^\alpha x_j^\beta$ ($\beta = 0$ if compartment j is outside environment)(see [2, 3, 5]) where $x_i(t)$ denotes the mass quantity of compartment i at time t and k_{ij} the exchange coefficient and α, β are constants characterizing the compartmental system. This is so-called the hypothesis of order polynomial exchange $(\alpha + \beta)$. The results of the bicompartmental system of polynomials are the result of the work of B. Hebri and Y. Cherruault (see [5]). The problem here is to determine the exchange coefficients between compartments by measuring of compartments the amount of the substance in a minimum number of compartments (not all the compartments). After injecting an amount of substance into compartment number one and waiting for a certain amount of time for the amount of substance to reach the second and third compartments. By measuring the amount of materials in the first and second compartments. The system is modeled using a nonlinear differential equation under the law of conservation of mass:

$$\frac{dx_i(t)}{dt} = \underbrace{\sum_{\substack{j=1\\ j\neq i}}^{n} k_{ji}x_j(t)}_{\sum \textbf{quantities entering}} - \underbrace{\sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}x_i(t)}_{\sum \textbf{quantities leaving}}$$

To be approximated to a linear differential equation using Taylor's formula, the linearization method is considered a stable method.

2 Definitions and notations

We consider the nonlinear tri-compartmental catenary system of polynomial type, namely $(S_{\mathrm{NL}}^{(P)})$, shown in figure (1).

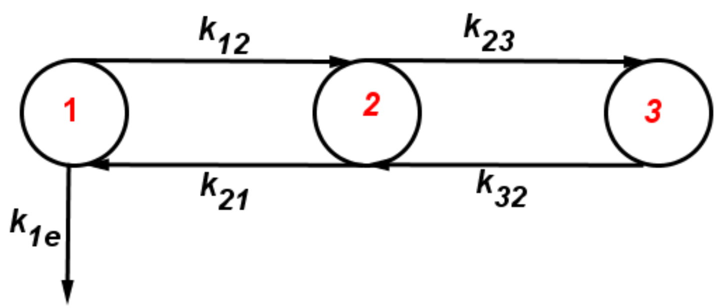

Figure 1: $\left(S_{\mathrm{NL}}^{(P)}\right)$: Nonlinear tri-compartimental catenary system.

The mass balance principle in each compartment leads to nonlinear differential equations (see [2]). The identification is done by exiting the system with and instantaneous injection of substance quantity a in the first compartment.
Thus we can say that the tri-compartmental catenary system is governed by the following differential system with initial condition:

$$\begin{cases} x_1'(t) = k_{21}x_2^{\alpha}(t)x_1^{\beta}(t) - k_{12}x_1^{\alpha}(t)x_2^{\beta}(t) - k_{1e}x_1^{\alpha}(t) \\ x_2'(t) = k_{12}x_1^{\alpha}(t)x_2^{\beta}(t) + k_{32}x_3^{\alpha}(t)x_2^{\beta}(t) - \left(k_{21}x_1^{\beta}(t) + k_{23}x_3^{\beta}(t)\right)x_2^{\alpha}(t) \\ x_3'(t) = k_{23}x_2^{\alpha}(t)x_3^{\beta}(t) - k_{32}x_3^{\alpha}(t)x_2^{\beta}(t) \\ x_1(0) = a \\ x_2(0) = 0 \\ x_3(0) = 0 \end{cases} \tag{1}$$

we note :

$$\begin{array}{rcl} X : [0,+\infty[& \longrightarrow & \mathbb{R}^3 \\ t & \longrightarrow & X^T(t) = (x_1(t), x_2(t), x_3(t)) \end{array}$$

The state function of the tri-compartmental catenary system $(S_{\mathrm{NL}}^{(P)})$, is:

$$\begin{array}{rcl} F:\mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x_1,x_2,x_3) & \longrightarrow & F(x_1,x_2,x_3) = (f_1(x_1,x_2,x_3), f_2(x_1,x_2,x_3), f_3(x_1,x_2,x_3)) \end{array}$$

such that:

$$\begin{cases} f_1(x_1,x_2,x_3) = k_{21}x_2^{\alpha}x_1^{\beta} - k_{12}x_1^{\alpha}x_2^{\beta} - k_{1e}x_1^{\alpha} \\ f_2(x_1,x_2,x_3) = k_{12}x_1^{\alpha}x_2^{\beta} + k_{32}x_3^{\alpha}x_2^{\beta} - \left(k_{21}x_1^{\beta} + k_{23}x_3^{\beta}\right)x_2^{\alpha} \\ f_3(x_1,x_2,x_3) = k_{23}x_2^{\alpha}x_3^{\beta} - k_{32}x_3^{\alpha}x_2^{\beta} \end{cases}$$

With these notations we can write the differential system (1) under the vectorial form:

$$\begin{cases} X'(t) = F^T(X^T(t)) \\ \\ X(0) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \end{cases} \tag{2}$$

3 Preliminary study

The partial derivatives of the function F being:

$$\begin{cases} \dfrac{\partial f_1}{\partial x_1}(x_1,x_2,x_3) = \beta k_{21}x_2^{\alpha}x_1^{\beta-1} - \alpha k_{12}x_1^{\alpha-1}x_2^{\beta} - \alpha k_{1e}x_1^{\alpha-1} \\ \\ \dfrac{\partial f_1}{\partial x_2}(x_1,x_2,x_3) = \alpha k_{21}x_2^{\alpha-1}x_1^{\beta} - \beta k_{12}x_1^{\alpha}x_2^{\beta-1} \\ \\ \dfrac{\partial f_1}{\partial x_3}(x_1,x_2,x_3) = 0 \\ \\ \dfrac{\partial f_2}{\partial x_1}(x_1,x_2,x_3) = \alpha k_{12}x_1^{\alpha-1}x_2^{\beta} - \beta k_{21}x_1^{\beta-1}x_2^{\alpha} \\ \\ \dfrac{\partial f_2}{\partial x_2}(x_1,x_2,x_3) = \beta k_{12}x_1^{\alpha}x_2^{\beta-1} + \beta k_{32}x_3^{\alpha}x_2^{\beta-1} - \alpha\left(k_{21}x_1^{\beta} + k_{23}x_3^{\beta}\right)x_2^{\alpha-1} \\ \\ \dfrac{\partial f_2}{\partial x_3}(x_1,x_2,x_3) = \alpha k_{32}x_3^{\alpha-1}x_2^{\beta} - k_{23}x_3^{\beta-1}x_2^{\alpha} \\ \\ \dfrac{\partial f_3}{\partial x_1}(x_1,x_2,x_3) = 0 \\ \\ \dfrac{\partial f_3}{\partial x_2}(x_1,x_2,x_3) = \alpha k_{23}x_2^{\alpha-1}x_3^{\beta} - \beta k_{32}x_3^{\alpha}x_2^{\beta-1} \\ \\ \dfrac{\partial f_3}{\partial x_3}(x_1,x_2,x_3) = \beta k_{23}x_2^{\alpha}x_3^{\beta-1} - \alpha k_{32}x_3^{\alpha-1}x_2^{\beta} \end{cases} \tag{3}$$

The function F is differentiable in all point (x_1, x_2, x_3) such that $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 \neq 0$ for all $\alpha > 0$ and all $\beta > 0$, and the Jacobain matrix is given by:

$$\left(DF_{(x_1,x_2,x_3)}\right) =$$

$$= \begin{pmatrix} -g_1(x_1,x_2,x_3) - \alpha k_{1e} x_1^{\alpha-1} & g_2(x_1,x_2,x_3) & 0 \\ g_1(x_1,x_2,x_3) & -g_2(x_1,x_2,x_3) - g_3(x_1,x_2,x_3) & g_4(x_1,x_2,x_3) \\ 0 & g_3(x_1,x_2,x_3) & -g_4(x_1,x_2,x_3) \end{pmatrix}$$

with:

$$\begin{cases} g_1(x_1, x_2, x_3 = \alpha k_{12} x_1^{\alpha-1} x_2^{\beta} - \beta k_{21} x_2^{\alpha} x_1^{\beta-1} \\ g_2(x_1,x_2,x_3) = \alpha k_{21} x_2^{\alpha-1} x_1^{\beta} - \beta k_{12} x_1^{\alpha} x_2^{\beta-1} \\ g_3(x_1,x_2,x_3) = \alpha k_{23} x_2^{\alpha-1} x_3^{\beta} - \beta k_{32} x_3^{\alpha} x_2^{\beta-1} \\ g_4(x_1,x_2,x_3) = \alpha k_{32} x_3^{\alpha-1} x_2^{\beta} - \beta k_{23} x_2^{\alpha} x_3^{\beta-1} \end{cases}$$

For the linearization of the system (2) we apply the Taylor formula in the neighborhood of the initial condition $(a, 0, 0)$.

Remark 1. *1. F is not differentiable in $(a, 0, 0)$ if $\alpha < 1$ or $\beta < 1$.*

2. If $\alpha \geq 1$ and $\beta \geq 1$, F is differentiable in $(a, 0, 0)$.

The Taylor formula applied in neighborhood of $(a, 0, 0)$ leads to:

$$F^T(x_1, x_2, x_3) =$$

$$= F^T(a,0,0) + (DF)_{(a,0,0)} (x_1 - a, x_2, x_3)^T + \left(D^2F\right)_{(x_{\theta 1}, x_{\theta 2}, x_{\theta 3})} (x_1 - a, x_2, x_3)^2$$

with: $(x_{\theta 1}, x_{\theta 2}, x_{\theta 3}) = (x_1 + \theta (x_1 - a), x_2 + \theta . x_2, x_3 + \theta . x_3)$ with $|\theta| < 1$.
For t sufficiently small, Our aim is to approach the differential system (1) on $[0, t_0]$ by the following linear differential system:

$$Z'(t) = F^T(a,0,0) + (DF)_{(a,0,0)} (z_1 - a, z_2, z_3)^T \tag{4}$$

with

$$(DF)_{(a,0,0)} = \begin{pmatrix} \alpha k_{1e} a^{\alpha-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We remark that this matrix is not well adapted to make an approximation of the exchange coefficients, so we introduce a temporization procedure. First we choose a time $t^* > 0$ close to 0 and consider the differential system witch governess the tri-compartmental catenary system, represented by the figure (1), at time t^*

$$\begin{cases} x_1'(t) = k_{21} x_2^{\alpha}(t) x_1^{\beta}(t) - k_{12} x_1^{\alpha}(t) x_2^{\beta}(t) - k_{1e} x_1^{\alpha}(t) \\ x_2'(t) = k_{12} x_1^{\alpha}(t) x_2^{\beta}(t) + k_{32} x_3^{\alpha}(t) x_2^{\beta}(t) - \left(k_{21} x_1^{\beta}(t) + k_{23} x_3^{\beta}(t)\right) x_2^{\alpha}(t) \\ x_3'(t) = k_{23} x_2^{\alpha}(t) x_3^{\beta}(t) - k_{32} x_3^{\alpha}(t) x_2^{\beta}(t) \\ x_1(t^*) = a_* \\ x_2(t^*) = b \\ x_3(t^*) = c \end{cases} \tag{5}$$

That we can write in the following equivalent vectorial form:

$$\begin{cases} X'(t)) = F^T\left(X^T(t)\right) \\ X^T(t^*) = (a_*, b, c) \end{cases} \tag{6}$$

The Taylor formula on the interval $[t^*, t_0]$ is given by:

$$F^T(x_1,x_2,x_3) = F^T(a_*,b,c) + (DF)_{(a_*,b,c)}(x_1 - a_*, x_2 - b, x_3 - c)^T$$

$$+ \left(D^2F\right)_{(x_{\theta 1},x_{\theta 2},x_{\theta 3})}(x_1 - a_*, x_2 - b, x_3 - c)^2$$

with: $(x_{\theta 1}, x_{\theta 2}, x_{\theta 3}) = (x_1 + \theta(x_1 - a_*), x_2 + \theta(x_2 - b), x_3 + \theta(x_3 - c))$ with $|\theta| < 1$.
and:

$$\left(DF_{(a_*,b,c)}\right) = \begin{pmatrix} -g_1(a_*,b,c) - \alpha k_{1e} a_*^{\alpha-1} & g_2(a_*,b,c) & 0 \\ g_1(a_*,b,c) & -g_2(a_*,b,c) - g_3(a_*,b,c) & g_4(a_*,b,c) \\ 0 & g_3(a_*,b,c) & -g_4(a_*,b,c) \end{pmatrix}$$

such that:

$$\begin{cases} g_1(a_*,b,c) = \alpha k_{12} a_*^{\alpha-1} b^\beta - \beta k_{21} b^\alpha a_*^{\beta-1} \\ g_2(a_*,b,c) = \alpha k_{21} b^{\alpha-1} a_*^\beta - \beta k_{12} a_*^\alpha b^{\beta-1} \\ g_3(a_*,b,c) = \alpha k_{23} c^\beta b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^\alpha \\ g_4(a_*,b,c) = \alpha k_{32} b^\beta c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^\alpha \end{cases}$$

and

$$F^T(a_*,b,c) = \begin{pmatrix} k_{21} b^\alpha a_*^\beta - k_{12} a_*^\alpha b^\beta - k_{1e} a_*^\alpha \ ; \\ k_{12} a_*^\alpha b^\beta + k_{32} c^\alpha b^\beta - \left(k_{21} a_*^\beta + k_{23} c^\beta\right) b^\alpha \ ; \\ k_{23} b^\alpha c^\beta - k_{32} c^\alpha b^\beta \end{pmatrix}$$

For t sufficiently small, we propose to approach the differential system (5) on $[t^*, t_0]$ by the following linear differential system:

$$Z'(t) = F^T(a_*,b,c) + (DF)_{(a_*,b,c)}(z_1(t) - a_*, z_2(t) - b, z_3(t) - c)^T \tag{7}$$

We pose:

$$\begin{cases} g_1(a_*,b,c) = g_1^* \\ g_2(a_*,b,c) = g_2^* \\ g_3(a_*,b,c) = g_3^* \\ g_4(a_*,b,c) = g_4^* \end{cases}$$

and

$$\begin{cases} f_1^* = k_{21} b^\alpha a_*^\beta - k_{12} a_*^\alpha b^\beta - k_{1e} a_*^\alpha \\ f_2^* = k_{12} a_*^\alpha b^\beta + k_{32} c^\alpha b^\beta - \left(k_{21} a_*^\beta + k_{23} c^\beta\right) b^\alpha \\ f_3^* = k_{23} b^\alpha c^\beta - k_{32} c^\alpha b^\beta \end{cases}$$

We can prove that there exists γ, δ and ω such that:

$$(DF_{(a_*,b,c)}). \begin{pmatrix} \gamma \\ \delta \\ \omega \end{pmatrix} = F^T(a_*,b,c)$$

indeed

$$\begin{pmatrix} -g_1^* - \alpha k_{1e} a_*^{\alpha-1} & g_2^* & 0 \\ g_1^* & -g_2^* - g_3^* & g_4^* \\ 0 & g_3^* & -g_4^* \end{pmatrix} \cdot \begin{pmatrix} \gamma \\ \delta \\ \omega \end{pmatrix} = \begin{pmatrix} f_1^* \\ f_2^* \\ f_3^* \end{pmatrix}$$

Or:

$$\begin{cases} -g_1^*\gamma - \alpha k_{1e} a_*^{\alpha-1}\gamma + g_2^*\delta = f_1^* \\ g_1^*\gamma - (g_2^* + g_3^*)\delta + g_4^*\omega = f_2^* \\ g_3^*\delta - g_4^*\omega = f_3^* \end{cases}$$

then:

$$\begin{cases} \gamma = \dfrac{-(f_1^* + f_2^* + f_3^*)}{\alpha k_{1e} a_*^{\alpha-1}} = \dfrac{a_*}{\alpha} \\ \\ \delta = \dfrac{g_1^*\gamma - f_3^* - f_2^*}{g_2^*} \\ \\ \omega = \dfrac{g_3^*\delta - f_3^*}{g_4^*} \end{cases}$$

Therefore we consider the linear differential system:

$$Z'(t) = (DF_{(a_*,b,c)}). \begin{pmatrix} z_1(t) - a_* + \gamma \\ z_2(t) - b + \delta \\ z_3(t) - c + \omega \end{pmatrix}$$

The change of the state function of the tri-compartmental catenary system:

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} z_1(t) - a_* + \gamma \\ z_2(t) - b + \delta \\ z_3(t) - c + \omega \end{pmatrix} \tag{8}$$

permits to reduce the system (6) to the canonical form:

$$\begin{cases} Y'(t) = (DF)_{(a_*,b,c)} .Y(t) \\ Y^T(t^*) = (\gamma, \delta, \omega) \end{cases} \tag{9}$$

The matrix $(DF)_{(a_*,b,c)}$ has the general form of a compartmental matrix, so to this matrix we can associate "formally" the compartmental linear system that we will note $\left(S_{lin}^{(TP)}\right)$ represented by the following figure:

with:

$$\begin{cases} p_{12} = \alpha k_{12} a_*^{\alpha-1} b^\beta - \beta k_{21} b^\alpha a_*^{\beta-1} \\ p_{21} = \alpha k_{21} b^{\alpha-1} a_*^\beta - \beta k_{12} a_*^\alpha b^{\beta-1} \\ p_{23} = \alpha k_{23} c^\beta b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^\alpha \\ p_{32} = \alpha k_{32} b^\beta c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^\alpha \\ p_{31} = p_{13} = 0 \\ p_{1e} = \alpha k_{1e} a_*^{\alpha-1} \end{cases}$$

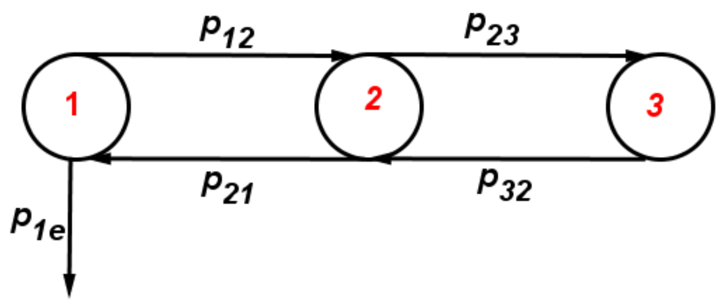

Figure 2: $\left(S_{lin}^{(TP)}\right)$ Linear model approximation.

Proposition 3.1. *The real numbers a_*, b and c such that*

$$p_{12} > 0 \qquad p_{21} > 0 \qquad p_{23} > 0 \qquad p_{32} > 0$$

exist if and only if $\alpha > \beta > 1$.

Proof. Knowing that:

$$\left\{ \begin{array}{l} p_{12} > 0 \\ p_{21} > 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \alpha k_{12} a_*^{\alpha} - \beta k_{21} b^{\alpha-\beta} a_*^{\beta} > 0 \\ \alpha k_{21} b^{\alpha-\beta} a_*^{\beta} - \beta k_{12} a_*^{\alpha} > 0 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} p_{23} > 0 \\ p_{32} > 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \alpha k_{23} b^{\alpha} - \beta k_{32} b^{\beta} c^{\alpha-\beta} > 0 \\ \alpha k_{32} b^{\beta} c^{\alpha-\beta} - \beta k_{23} b^{\alpha} > 0 \end{array} \right.$$

Let $x = k_{12} a_*^{\alpha}$ and $y = k_{21} b^{\alpha-\beta}$ and $z = \alpha k_{23} b^{\alpha}$ and $w = \beta k_{32} b^{\beta} c^{\alpha-\beta}$ $(x > 0 \ \text{ and } \ y > 0 \ \text{ and } \ z > 0 \ \text{ and } \ w > 0)$

$$\left\{ \begin{array}{l} p_{12} > 0 \\ p_{21} > 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \alpha x - \beta y > 0 \\ \alpha y - \beta x > 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} p_{23} > 0 \\ p_{32} > 0 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} \alpha z - \beta w > 0 \\ \alpha w - \beta z > 0 \end{array} \right.$$

If $\alpha \leq \beta$ the solutions set:

$$\left\{ \begin{array}{l} \alpha x - \beta y > 0 \\ \alpha y - \beta x > 0 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \alpha z - \beta w > 0 \\ \alpha w - \beta z > 0 \end{array} \right.$$

is empty.

If $\alpha > \beta > 1$ the solutions set :

$$\left\{ \begin{array}{l} \alpha x - \beta y > 0 \\ \alpha y - \beta x > 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \alpha z - \beta w > 0 \\ \alpha w - \beta z > 0 \end{array} \right.$$

is not empty.

□

3 Calculation of the exchange coefficients $\{p_{ij}/\ ij=1,2,3 \quad i \neq j\}$ of the system $\left(S_{lin}^{(TP)}\right)$ and the excretion coefficient p_{1e}

Note the compartmental matrix of the linear model $\left(S_{lin}^{(TP)}\right)$ by:

$$A = \begin{pmatrix} -p_{12}-p_{1e} & p_{21} & 0 \\ p_{12} & -p_{21}-p_{23} & p_{32} \\ 0 & p_{23} & -p_{32} \end{pmatrix}$$

The matrix A being tridiagonal and compartmental, its eigenvalues noted λ_i $\{i \in \{1,2,3\}\}$ are real, distinct and strictly negative. The general solution of the system is written in the form

$$y_j(t) = \sum_{i=1}^{n} \beta_i^j exp\left(\lambda_i t\right) \quad \forall i \in \{1,2,3\}$$

where β_i^j $(i \in \{1,2,3\})$ is the Jacolumn of the matrix B of the elementary masses, associated with the i compartment.

The measurements made on the first and the second compartment make the minimization of the functional J introduced by Y. Cherruault [4] possible:

$$J\left(\beta_k^i, \lambda_k, 1 \leq i \leq 2, 1 \leq k \leq 3\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{2}\left(x_i(t_j) - \sum_{k=1}^{3}\beta_k^i e^{\lambda_k t_j}\right)^2\right) \tag{10}$$

we put:

$$\min J\left(\beta_k^i, \lambda_k, 1 \leq i \leq 2, 1 \leq k \leq 3\right) = J\left(\beta_k^{i*}, \lambda_k^*, 1 \leq i \leq 2, 1 \leq k \leq 3\right)$$

The functions $\varphi_i, \quad i \in \{1,2,3\}$ defined by:

$$\varphi_i\left(t\right) = \exp\left(\lambda_i^* t\right) \quad ,\forall t \geq t^*, \quad \forall i \in \{1,2,3\}$$

being linearly independent we can conclude that for every integer i in $\{1,2,3\}$ we have:

$$\lambda i^* \beta_i^{1*} = -p_{12}\beta_i^{1*} - p_{1e}\beta_i^{1*} + p_{21}\beta_i^{2*} \tag{11}$$

$$\lambda_i^* \beta_i^{2*} = p_{12}\beta_i^{1*} - p_{21}\beta_i^{2*} - p_{23}\beta_i^{2*} + p_{32}\beta_i^{3} \tag{12}$$

$$\lambda_i^* \beta_i^{3} = p_{23}\beta_i^{2*} - p_{32}\beta_i^{3} \tag{13}$$

From the relationship (11) for $i=1$ and $i=2$ we have:

$$\begin{cases} \lambda_1^*\beta_1^{1*} = -p_{12}\beta_1^{1*} - p_{1e}\beta_1^{1*} + p_{21}\beta_1^{2*} \\ \lambda_2^*\beta_2^{1*} = -p_{12}\beta_2^{1*} - p_{1e}\beta_2^{1*} + p_{21}\beta_2^{2*} \end{cases}$$

we conclude then that if $\beta_1^{2*}\beta_2^{1*} - \beta_2^{2*}\beta_1^{1*} \neq 0$:

$$p_{21} = \frac{\left(\lambda_1^* - \lambda_2^*\right)\beta_1^{1*}\beta_2^{1*}}{\beta_1^{2*}\beta_2^{1*} - \beta_2^{2*}\beta_1^{1*}} \quad . \tag{14}$$

We recall that:

$$x_1'^* = k_{21} b^\alpha a_*^\beta - k_{12} a_*^\alpha b^\beta - k_{1e} a_*^\alpha$$

and

$$x_1'^* = y_1'(t^*) = \lambda_1^* \beta_1^{1*} e^{\lambda_1^* t^*} + \lambda_2^* \beta_2^{1*} e^{\lambda_2^* t^*} + \lambda_3^* \beta_3^{1*} e^{\lambda_3^* t^*}$$

we recall too that:

$$\begin{cases} p_{12} = \alpha k_{12} a_*^{\alpha-1} b^\beta - \beta k_{21} b^\alpha a_*^{\beta-1} \\ \\ p_{21} = \alpha k_{21} b^{\alpha-1} a_*^\beta - \beta k_{12} a_*^\alpha b^{\beta-1} \end{cases}$$

therefore:

$$\begin{cases} \dfrac{a_*}{\alpha} p_{12} = k_{12} a_*^\alpha b^\beta - \dfrac{\beta}{\alpha} k_{21} b^\alpha a_*^\beta \\ \\ \dfrac{b}{\alpha} p_{21} = k_{21} b^\alpha a_*^\beta - \dfrac{\beta}{\alpha} k_{12} a_*^\alpha b^\beta \end{cases}$$

consequently:

$$\frac{a_*}{\alpha} p_{12} - \frac{b}{\alpha} p_{21} = k_{12} a_*^\alpha b^\beta - k_{21} b^\alpha a_*^\beta - \frac{\beta}{\alpha} \left(k_{21} b^\alpha a_*^\beta - k_{12} a_*^\alpha b^\beta \right)$$

$$\frac{a_*}{\alpha} p_{12} - \frac{b}{\alpha} p_{21} = -x_1'^*) - k_{1e} a_*^\alpha - \frac{\beta}{\alpha} (x_1'^*) + k_{1e} a_*^\alpha)$$

and we have:

$$k_{1e} a_*^\alpha = \frac{a_*}{\alpha} p_{1e}$$

therefore:

$$\frac{a_*}{\alpha} p_{12} - \frac{b}{\alpha} p_{21} = x_1'^*) \left(-1 - \frac{\beta}{\alpha} \right) + \frac{a_*}{\alpha} p_{1e} \left(-1 - \frac{\beta}{\alpha} \right)$$

then:

$$\frac{a_*}{\alpha} p_{12} + \frac{a_*}{\alpha} p_{1e} \left(1 + \frac{\beta}{\alpha} \right) = \frac{b}{\alpha} p_{21} + x_1'^*) \left(-1 - \frac{\beta}{\alpha} \right)$$

the relationship (11) for $i = 3$ gives:

$$\beta_3^{1*} p_{12} + p_{1e} \beta_3^{1*} = p_{21} \beta_3^{2*} - \lambda_3^* \beta_3^{1*}$$

therefore:

$$\begin{pmatrix} \dfrac{a_*}{\alpha} & \dfrac{a_*}{\alpha} \left(1 + \dfrac{\beta}{\alpha} \right) \\ \beta_3^{1*} & \beta_3^{1*} \end{pmatrix} \begin{pmatrix} p_{12} \\ p_{1e} \end{pmatrix} = \begin{pmatrix} \dfrac{b}{\alpha} p_{21} + x_1'^*) \left(-1 - \dfrac{\beta}{\alpha} \right) \\ p_{21} \beta_3^{2*} - \lambda_3^* \beta_3^{1*} \end{pmatrix}$$

by putting:

$$M = \begin{pmatrix} \dfrac{a_*}{\alpha} & \dfrac{a_*}{\alpha} \left(1 + \dfrac{\beta}{\alpha} \right) \\ \beta_3^{1*} & \beta_3^{1*} \end{pmatrix}$$

$$M_1 = \begin{pmatrix} \frac{b}{\alpha}p_{21} + x_1'^{*}) \left(-1 - \frac{\beta}{\alpha}\right) & \frac{a_*}{\alpha}\left(1 + \frac{\beta}{\alpha}\right) \\ p_{21}\beta_3^{2*} - \lambda_3^* \beta_3^{1*} & \beta_3^{1*} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} \frac{a_*}{\alpha} & \frac{b}{\alpha}p_{21} + x_1'^{*}) \left(-1 - \frac{\beta}{\alpha}\right) \\ \beta_3^{1*} & p_{21}\beta_3^{2*} - \lambda_3^* \beta_3^{1*} \end{pmatrix}$$

and if $\beta_3^{1*} \neq 0$, we obtain:

$$\begin{cases} p_{12} = \dfrac{\det M_1}{\det M} \\ \\ p_{1e} = \dfrac{\det M_2}{\det M} \end{cases} \tag{15}$$

Now, if we add up the relationships (11), (12) and (13) member to member, we get:

$$\beta_i^3 = \beta_i^{3*} = -\beta_i^{1*} - \beta_i^{2*} - p_{1e}\frac{\beta_i^{1*}}{\lambda_i^*} \quad \forall i \in \{1, 2, 3\}\,. \tag{16}$$

The relationship (13) for $i = 1$ and $i = 2$ gives:

$$\begin{cases} \lambda_1^*\beta_1^{3*} = p_{23}\beta_1^{2*} - p_{32}\beta_1^{3*} \\ \lambda_2^*\beta_2^{3*} = p_{23}\beta_2^{2*} - p_{32}\beta_2^{3*} \end{cases} \tag{17}$$

if $\beta_1^{2*}\beta_2^{3*} - \beta_2^{2*}\beta_1^{3*} \neq 0$, we have:

$$p_{23} = \frac{\left(\lambda_1^* - \lambda_2^*\right)\beta_1^{3*}\beta_2^{3*}}{\beta_1^{2*}\beta_2^{3*} - \beta_2^{2*}\beta_1^{3*}} \tag{18}$$

and

$$p_{32} = -\frac{\lambda_1^*\beta_1^{3*}\beta_2^{2*} - \lambda_2^*\beta_2^{3*}\beta_1^{2*}}{\beta_1^{3*}\beta_2^{2*} - \beta_2^{3*}\beta_1^{2*}} \tag{19}$$

The exchange coefficients $p_{1e}, p_{12}, p_{21}, p_{23},$ and p_{32} are then identify. We denote:

$$\begin{cases} p_{1e} = \nu_1^* \\ p_{12} = \nu_2^* \\ p_{21} = \nu_3^* \\ p_{23} = \nu_4^* \\ p_{32} = \nu_5^* \end{cases}$$

3.1 Calculation of the initial condition c

Proposition 3.1. *The initial condition c is given by:*

$$c = \frac{1}{\nu_5^*}\left[(\alpha + \beta)\left(\lambda_1^*\beta_1^{3*}e^{\lambda_1^* t^*} + \lambda_2^*\beta_2^{3*}e^{\lambda_2^* t^*} + \lambda_3^*\beta_2^{3*}e^{\lambda_3^* t^*}\right) + b\nu_4^*\right]$$

Proof. We have

$$x_3'(t^*) = y_3'(t^*)$$

therefore:

$$k_{32} b^\beta c^\alpha - k_{23} c^\beta b^\alpha = \lambda_1^* \beta_1^{3*} e^{\lambda_1^* t^*} + \lambda_2^* \beta_2^{3*} e^{\lambda_2^* t^*} + \lambda_3^* \beta_2^{3*} e^{\lambda_3^* t^*}$$

and we have

$$\begin{aligned} \nu_4^* &= \alpha k_{23} c^\beta b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^\alpha \\ \nu_5^* &= \alpha k_{32} b^\beta c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^\alpha \end{aligned}$$

so:

$$c\nu_5^* - b\nu_4^* = (\alpha+\beta)\left(k_{32} b^\beta c^\alpha - k_{23} c^\beta b^\alpha\right)$$

consequently:

$$c = \frac{1}{\nu_5^*}\left[(\alpha+\beta)\left(k_{32} b^\beta c^\alpha - k_{23} c^\beta b^\alpha\right) + b\nu_4^*\right]$$

Finally we have:

$$c = \frac{1}{\nu_5^*}\left[(\alpha+\beta)\left(\lambda_1^* \beta_1^{3*} e^{\lambda_1^* t^*} + \lambda_2^* \beta_2^{3*} e^{\lambda_2^* t^*} + \lambda_3^* \beta_2^{3*} e^{\lambda_3^* t^*}\right) + b\nu_4^*\right]$$

□

4 Calculation of the excretion coefficient k_{1e} and the exchange coefficients $\{k_{ij}/ \;\; i,j = 1,2,3 \;\; i \neq j\}$ of the system $\left(S_{\mathrm{NL}}^{(P)}\right)$

Proposition 4.1. *Let* $p^* = \begin{pmatrix} \beta_1^{1*} & \beta_1^{2*} & \beta_1^{3*} \\ \beta_2^{1*} & \beta_2^{2*} & \beta_2^{3*} \\ \beta_3^{1*} & \beta_3^{2*} & \beta_3^{3*} \end{pmatrix}$ *the partial measurement matrix associated to the system* $\left(S_{lin}^{(TP)}\right)$ *identified by (9) if* $\alpha > \beta > 1$ *then the nonlinear system* $\left(S_{NL}^{(p)}\right)$ *is identified. Furthermore we have:*

$$k_{1e} = \frac{\det M_2}{a_*^{\alpha-1} \det M}$$

$$k_{12} = \frac{a_* \alpha \nu_2^* + b\beta\nu_3^*}{(\alpha^2-\beta^2) b^\beta a_*^\alpha}; \quad k_{21} = \frac{a_* \beta \nu_2^* + b\nu_3^* \alpha}{(\alpha^2-\beta^2) b^\alpha a_*^\beta}$$

Proposition 4.2.

$$k_{23} = \frac{b\alpha\nu_4^* + c\beta\nu_5^*}{(\alpha^2-\beta^2) c^\beta b^\alpha} \qquad ; \qquad k_{32} = \frac{b\beta\nu_4^* + c\nu_5^*\alpha}{(\alpha^2-\beta^2) c^\alpha b^\beta} \quad .$$

Proof. We proved that:

$$p_{1e} = \frac{\det M_2}{\det M}$$

and we have:

$$k_{1e} = \frac{p_{1e}}{a_*^{\alpha-1}}$$

consequently:

$$k_{1e} = \frac{\det M_2}{a_*^{\alpha-1}\det M}$$

We recall firstly that:

$$\begin{aligned} p_{12} = \nu_2^* = \alpha k_{12} b^\beta a_*^{\alpha-1} - \beta k_{21} a_*^{\beta-1} b^\alpha \\ p_{21} = \nu_3^* = \alpha k_{21} a_*^\beta b^{\alpha-1} - \beta k_{12} b^{\beta-1} a_*^\alpha \end{aligned}$$

a linear combination of them gives:

$$a_* \alpha \nu_2^* + b\beta\nu_3^* = k_{12} b^\beta a_*^\alpha (\alpha^2 - \beta^2)$$

so:

$$k_{12} = \frac{a_*\alpha\nu_2^* + b\beta\nu_3^*}{(\alpha^2 - \beta^2) b^\beta a_*^\alpha}$$

another linear combination of them gives:

$$a_*\beta\nu_2^* + b\alpha\nu_3^* = (\alpha^2 - \beta^2) b^\alpha a_*^\beta k_{21}$$

then:

$$k_{21} = \frac{a_*\beta\nu_2^* + b\nu_3^*\alpha}{(\alpha^2 - \beta^2) b^\alpha a_*^\beta} \quad .$$

We recall secondly that:

$$\begin{aligned} p_{23} = \nu_4^* = \alpha k_{23} c^\beta b^{\alpha-1} - \beta k_{32} b^{\beta-1} c^\alpha \\ p_{32} = \nu_5^* = \alpha k_{32} b^\beta c^{\alpha-1} - \beta k_{23} c^{\beta-1} b^\alpha \end{aligned}$$

therefore:

$$b\alpha\nu_4^* + c\beta\nu_5^* = k_{23} c^\beta b^\alpha (\alpha^2 - \beta^2)$$

so:

$$k_{23} = \frac{b\alpha\nu_4^* + c\beta\nu_5^*}{(\alpha^2 - \beta^2) c^\beta b^\alpha}$$

and

$$b\beta\nu_4^* + c\alpha\nu_5^* = (\alpha^2 - \beta^2) c^\alpha b^\beta k_{32}$$

then:

$$k_{32} = \frac{b\beta\nu_4^* + c\nu_5^*\alpha}{(\alpha^2 - \beta^2) c^\alpha b^\beta} \quad .$$

□

5 Stability of the linearization method

Notation 1. *We note the exchange coefficients of a real linear tri-compartmental catenary system by:*

$$\begin{cases} \overline{p}_{12} = \vartheta_1^* \\ \overline{p}_{21} = \vartheta_2^* \\ \overline{p}_{23} = \vartheta_3^* \\ \overline{p}_{32} = \vartheta_4^*, \end{cases}$$

and the coefficient exchange of a real nonlinear tri-compartmental catenary system by $\overline{k}_{12}$; $\overline{k}_{21}$; $\overline{k}_{23}$; $\overline{k}_{32}$, *and note that* ε_1 ; ε_2 ; ε_3 *and* ε_4 *errors made on the calculation of* p_{12} , p_{21} , p_{23} *and* p_{32} *respectively*

Proposition 5.1. *The exchange coefficients of the nonlinear polynomial system can be approached by:*

$$\begin{cases} \overline{k}_{12} = \dfrac{\alpha\vartheta_1^* a_* + \beta\vartheta_2^* b}{(\alpha^2-\beta^2)a_*^\alpha b^\beta} \\ \overline{k}_{21} = \dfrac{\alpha\vartheta_2^* b + \beta\vartheta_1^* a_*}{(\alpha^2-\beta^2)a_*^\beta b^\alpha} \end{cases} \tag{20}$$

and

$$\begin{cases} \overline{k}_{23} = \dfrac{\alpha\vartheta_3^* b + \beta\vartheta_4^* c}{(\alpha^2-\beta^2)b^\alpha c^\beta} \\ \overline{k}_{32} = \dfrac{\alpha\vartheta_4^* c + \beta\vartheta_3^* b}{(\alpha^2-\beta^2)b^\beta c^\alpha} \end{cases} \tag{21}$$

Proposition 5.2. *which represent the respective approximations of* k_{12}, k_{21}, k_{23} *and* k_{32}. *More precisely:*

$$\begin{cases} |k_{12} - \overline{k}_{12}| \leq \max(\varepsilon_1, \varepsilon_2).\dfrac{\alpha a_* + \beta b}{(\alpha^2-\beta^2)a_*^\beta b^\alpha} \\ \\ |k_{21} - \overline{k}_{21}| \leq \max(\varepsilon_1, \varepsilon_2).\dfrac{\alpha b + \beta a_*}{(\alpha^2-\beta^2)a_*^\beta b^\alpha}, \end{cases} \tag{22}$$

and

$$\begin{cases} |k_{23} - \overline{k}_{23}| \leq \max(\varepsilon_3, \varepsilon_4).\dfrac{\alpha b + \beta c}{(\alpha^2-\beta^2)b^\beta c^\alpha} \\ \\ |k_{32} - \overline{k}_{32}| \leq \max(\varepsilon_3, \varepsilon_4).\dfrac{\alpha c + \beta b}{(\alpha^2-\beta^2)b^\beta c^\alpha} \end{cases} \tag{23}$$

Proof. ϑ_1^* being an approximation of p_{12} then there exists $\varepsilon_1^{'}(|\varepsilon_1^{'}| \leq \varepsilon_1)$ such that:

$$\vartheta_1^* + \varepsilon_1^{'} = \alpha k_{12} b^\beta a_*^{\alpha-1} - \beta k_{21} b^\alpha a_*^{\beta-1} \tag{24}$$

by following:

$$\alpha b^{-1} a_*(\vartheta_1^* + \varepsilon_1^{'}) = \alpha^2 k_{12} b^{\beta-1} a_*^\alpha - \alpha\beta k_{21} b^{\alpha-1} a_*^\beta \tag{25}$$

ϑ_2^* being an approximation of p_{21} then there exists $\varepsilon_2^{'}(|\varepsilon_2^{'}| \leq \varepsilon_2)$ such that:

$$\vartheta_2^* + \varepsilon_2^{'} = \alpha k_{21} b^{\alpha-1} a_*^\beta - \beta k_{12} b^{\beta-1} a_*^\alpha \tag{26}$$

by following:

$$\beta(\vartheta_2^* + \varepsilon_2^{'}) = \alpha\beta k_{21} b^{\alpha-1} a_*^{\beta} - \beta^2 k_{12} b^{\beta-1} a_*^{\alpha} \tag{27}$$

if we add up the two the relation (25) and (27) member to member, we get

$$\alpha b^{-1} a_*(\vartheta_1^* + \varepsilon_1^{'}) + \beta(\vartheta_2^* + \varepsilon_2^{'}) = \alpha^2 k_{12} b^{\beta-1} a_*^{\alpha} - \beta^2 k_{12} b^{\beta-1} a_*^{\alpha}$$

therefore:

$$k_{12} = \frac{\alpha a_*(\vartheta_1^* + \varepsilon_1^{'}) + \beta b(\vartheta_2^* + \varepsilon_2^{'})}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}}.$$

We can see too that the relationship (28) and (29) are equivalent respectively to:

$$\beta a_* b^{-1}(\vartheta_1^* + \varepsilon_1^{'}) = \beta\alpha k_{12} b^{\beta-1} a_*^{\alpha} - \beta^2 k_{21} b^{\alpha-1} a_*^{\beta} \tag{28}$$

and

$$\alpha(\vartheta_2^* + \varepsilon_2^{'}) = \alpha^2 k_{21} b^{\alpha-1} a_*^{\beta} - \beta\alpha k_{12} b^{\beta-1} a_*^{\alpha} \tag{29}$$

by adding the two relations (28) and (29) member to member, we obtain:

$$\beta a_* b^{-1}(\vartheta_1^* + \varepsilon_1^{'}) + \alpha(\vartheta_2^* + \varepsilon_2^{'}) = \alpha^2 k_{21} b^{\alpha-1} a_*^{\beta} - \beta^2 k_{21} b^{\alpha-1} a_*^{\beta}$$

therefore:

$$k_{21} = \frac{\beta a_*(\vartheta_1^* + \varepsilon_1^{'}) + b\alpha(\vartheta_2^* + \varepsilon_2^{'})}{(\alpha^2 - \beta^2) b^{\alpha} a_*^{\beta}}.$$

knowing that:

$$\overline{k}_{12} = \lim_{(\varepsilon_1^{'}, \varepsilon_2^{'}) \longrightarrow (0,0)} \frac{\alpha a_*(\vartheta_1^* + \varepsilon_1^{'}) + \beta b(\vartheta_2^* + \varepsilon_2^{'})}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}} = \frac{\alpha a_* \vartheta_1^* + \beta b \vartheta_2^*}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}}$$

and

$$\overline{k}_{21} = \lim_{(\varepsilon_1^{'}, \varepsilon_2^{'}) \longrightarrow (0,0)} \frac{\beta a_*(\vartheta_1^* + \varepsilon_1^{'}) + b\alpha(\vartheta_2^* + \varepsilon_2^{'})}{(\alpha^2 - \beta^2) b^{\alpha} a_*^{\beta}} = \frac{\beta a_* \vartheta_1^* + b\alpha \vartheta_2^*}{(\alpha^2 - \beta^2) b^{\alpha} a_*^{\beta}}$$

we can write that:

$$|k_{12} - \overline{k}_{12}| = \left| \frac{\alpha a_*(\vartheta_1^* + \varepsilon_1^{'}) + \beta b(\vartheta_2^* + \varepsilon_2^{'})}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}} - \frac{\alpha a_* \vartheta_1^* + b\beta \vartheta_2^*}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}} \right|$$

$$|k_{12} - \overline{k}_{12}| = \left| \frac{\alpha a_* \varepsilon_1^{'} + b\beta \varepsilon_2^{'}}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}} \right| \leq \left| \frac{(\alpha a_* + b\beta) \max\left(\varepsilon_1^{'}, \varepsilon_2^{'}\right)}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}} \right|$$

so:

$$|k_{12} - \overline{k}_{12}| \leq \frac{(\alpha a_* + b\beta)\, max\, (\varepsilon_1, \varepsilon_2)}{(\alpha^2 - \beta^2) b^{\beta} a_*^{\alpha}},$$

and

$$|k_{21} - \overline{k}_{21}| = \left| \frac{\beta a_*(\vartheta_1^* + \varepsilon_1^{'}) + \alpha b(\vartheta_2^* + \varepsilon_2^{'})}{(\alpha^2 - \beta^2) b^{\alpha} a_*^{\beta}} - \frac{\beta a_* \vartheta_1^* + b\alpha \vartheta_2^*}{(\alpha^2 - \beta^2) b^{\alpha} a_*^{\beta}} \right|$$

$$|k_{12} - \overline{k}_{12}| = \left| \frac{\beta a_* \varepsilon_1^{'} + b\alpha \varepsilon_2^{'}}{(\alpha^2 - \beta^2) b^{\alpha} a_*^{\beta}} \right|$$

$$|k_{12}-\overline{k}_{12}| \leq \left| \frac{(\beta a_* + b\alpha)\max\left(\varepsilon_1^{'},\varepsilon_2^{'}\right)}{(\alpha^2-\beta^2)b^{\alpha}a_*^{\beta}} \right|$$

so:

$$|k_{12}-\overline{k}_{12}| \leq \frac{(\beta a_* + b\alpha)\max\left(\varepsilon_1,\varepsilon_2\right)}{(\alpha^2-\beta^2)b^{\alpha}a_*^{\beta}}.$$

ϑ_3^* being an approximation of p_{23} there exists $\varepsilon_3^{'}(|\varepsilon_3^{'}| \leq \varepsilon_3)$ such that:

$$\vartheta_3^* + \varepsilon_3^{'} = \alpha k_{23}c^{\beta}b^{\alpha-1} - \beta k_{32}c^{\alpha}b^{\beta-1} \tag{30}$$

ϑ_4^* being an approximation of p_{32} there exists $\varepsilon_4^{'}(|\varepsilon_4^{'}| \leq \varepsilon_4)$ such that:

$$\vartheta_4^* + \varepsilon_4^{'} = \alpha k_{32}c^{\alpha-1}b^{\beta} - \beta k_{23}c^{\beta-1}b^{\alpha} \tag{31}$$

First step:

Multiplying the two members of the relation (30) by the number αbc^{-1}, we get:

$$\alpha c^{-1}b(\vartheta_3^* + \varepsilon_3^{'}) = \alpha^2 k_{23}c^{\beta-1}b^{\alpha} - \alpha\beta k_{32}c^{\alpha-1}b^{\beta} \tag{32}$$

by multiplying the two members of the relation (31) by the number β, we get:

$$\beta(\vartheta_4^* + \varepsilon_4^{'}) = \alpha\beta k_{32}c^{\alpha-1}b^{\beta} - \beta^2 k_{23}c^{\beta-1}b^{\alpha} \tag{33}$$

if we add up the relations (32) and (33) member to member, we get:

$$\alpha c^{-1}b(\vartheta_3^* + \varepsilon_3^{'}) + \beta(\vartheta_4^* + \varepsilon_4^{'}) = \alpha^2 k_{23}c^{\beta-1}b^{\alpha} - \beta^2 k_{23}c^{\beta-1}b^{\alpha}$$

by multiplying the two terms by c we will have:

$$\alpha b(\vartheta_3^* + \varepsilon_3^{'}) + \beta c(\vartheta_4^* + \varepsilon_4^{'}) = k_{23}(\alpha^2-\beta^2)c^{\beta}b^{\alpha}$$

therefore:

$$k_{23} = \frac{\alpha b(\vartheta_3^* + \varepsilon_3^{'}) + \beta c(\vartheta_4^* + \varepsilon_4^{'})}{(\alpha^2-\beta^2)c^{\beta}b^{\alpha}}.$$

Second step:

Multiplying the two members of the relation (30) by the number βbc^{-1}, we obtain:

$$\beta bc^{-1}(\vartheta_3^* + \varepsilon_3^{'}) = \beta\alpha k_{23}c^{\beta-1}b^{\alpha} - \beta^2 k_{32}c^{\alpha-1}b^{\beta} \tag{34}$$

by multiplying the two members of the relation (31) by the number α, we get:

$$\alpha(\vartheta_4^* + \varepsilon_4^{'}) = \alpha^2 k_{32}c^{\alpha-1}b^{\beta} - \beta\alpha k_{23}c^{\beta-1}b^{\alpha} \tag{35}$$

if we add up the relations (34) and (35) member to member, we obtain:

$$\beta bc^{-1}(\vartheta_3^* + \varepsilon_3^{'}) + \alpha(\vartheta_4^* + \varepsilon_4^{'}) = \alpha^2 k_{32}c^{\alpha-1}b^{\beta} - \beta^2 k_{32}c^{\alpha-1}b^{\beta}$$

by multiplying the two terms by c we will have:

$$\beta b(\vartheta_3^* + \varepsilon_3^{'}) + c\alpha(\vartheta_4^* + \varepsilon_4^{'}) = k_{32}(\alpha^2-\beta^2)c^{\alpha}b^{\beta}$$

therefore:

$$k_{32}=\frac{\beta b(\vartheta_3^*+\varepsilon_3^{'})+c\alpha(\vartheta_4^*+\varepsilon_4^{'})}{(\alpha^2-\beta^2)c^\alpha b^\beta}.$$

knowing that:

$$\overline{k}_{23}=\lim_{(\varepsilon_3^{'},\varepsilon_4^{'})\longrightarrow(0,0)}\frac{\alpha b(\vartheta_3^*+\varepsilon_3^{'})+\beta c(\vartheta_4^*+\varepsilon_4^{'})}{(\alpha^2-\beta^2)c^\beta b^\alpha}=\frac{\alpha b\vartheta_3^*+\beta c\vartheta_4^*}{(\alpha^2-\beta^2)c^\beta b^\alpha}$$

and

$$\overline{k}_{32}=\lim_{(\varepsilon_3^{'},\varepsilon_4^{'})\longrightarrow(0,0)}\frac{\beta b(\vartheta_3^*+\varepsilon_3^{'})+c\alpha(\vartheta_4^*+\varepsilon_4^{'})}{(\alpha^2-\beta^2)c^\alpha b^\beta}=\frac{\beta b\vartheta_3^*+c\alpha\vartheta_4^*}{(\alpha^2-\beta^2)c^\alpha b^\beta}$$

we can write:

$$|k_{23}-\overline{k}_{23}|=\left|\frac{\alpha b(\vartheta_3^*+\varepsilon_3^{'})+\beta c(\vartheta_4^*+\varepsilon_4^{'})}{(\alpha^2-\beta^2)c^\beta b^\alpha}-\frac{\alpha b\vartheta_3^*+c\beta\vartheta_4^*}{(\alpha^2-\beta^2)c^\beta b^\alpha}\right|$$

$$|k_{23}-\overline{k}_{23}|=\left|\frac{\alpha b\varepsilon_3^{'}+c\beta\varepsilon_4^{'}}{(\alpha^2-\beta^2)c^\beta b^\alpha}\right|\leq\left|\frac{(\alpha b+c\beta)\max\left(\varepsilon_3^{'},\varepsilon_4^{'}\right)}{(\alpha^2-\beta^2)c^\beta b^\alpha}\right|$$

so:

$$|k_{23}-\overline{k}_{23}|\leq\frac{(\alpha b+c\beta)\max\left(\varepsilon_3,\varepsilon_4\right)}{(\alpha^2-\beta^2)c^\beta b^\alpha},$$

and

$$|k_{32}-\overline{k}_{32}|=\left|\frac{\beta b(\vartheta_3^*+\varepsilon_3^{'})+\alpha c(\vartheta_4^*+\varepsilon_4^{'})}{(\alpha^2-\beta^2)c^\alpha b^\beta}-\frac{\beta b\vartheta_3^*+c\alpha\vartheta_4^*}{(\alpha^2-\beta^2)c^\alpha b^\beta}\right|$$

$$|k_{32}-\overline{k}_{32}|=\left|\frac{\beta b\varepsilon_3^{'}+c\alpha\varepsilon_4^{'}}{(\alpha^2-\beta^2)c^\alpha b^\beta}\right|\leq\left|\frac{(\beta b+c\alpha)\max\left(\varepsilon_3^{'},\varepsilon_4^{'}\right)}{(\alpha^2-\beta^2)c^\alpha b^\beta}\right|$$

so:

$$|k_{32}-\overline{k}_{32}|\quad\leq\frac{(\beta b+c\alpha)\max\left(\varepsilon_3,\varepsilon_4\right)}{(\alpha^2-\beta^2)c^\alpha b^\beta}.$$

□

Remark 1. *We have:*

$$p_{1e}=\overline{p}_{1e}\quad\textit{(see [7])}$$

So:

$$k_{1e}=\frac{\overline{p}_{1e}}{\alpha a_*^{\alpha-1}}.$$

6 Conclusion

1. The linear model associated to the non linear polynomial tri-compartmental catenary system of $(\alpha+\beta)$ order involves four important difficulties:

   The initial condition at time $t=0$ does not permit to give a complete information about the model $(S_{\mathrm{NL}}^{(P)})$. A temporization t^* is introduced to suppress this difficulty.

2. If this temporization is not modulated, the linear model is not necessarily real. We have shown that the measures done on the compartment 1 and on the compartment 2 permit to choose one measure at instant $t_{i1} = t^*$ such that we can develop a linearization method.

3. The nonhomogeneous condition $x_3(t^*) = c$ being unknown is identified form measures done on compartment 1 and on compartment 2.

4. The linearization method is stable.

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