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DETERMINATION OF A KERNEL IN A NONLOCAL PROBLEM FOR THE TIME-FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION

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Abstract In this work, we consider an inverse problem of determining the kernel of a fractional diffusion equation. The backward problem is the initial-boundary/value problem for this equation with non-local initial and homogeneous Dirichlet conditions. To determine the kernel, an overdetermination condition of the integral form is specified for the solution of the backward problem. Using the Fourier method and an ordinary fractional differential equation with a non-local boundary condition, the inverse problem is reduced to an equivalent problem. Further by using the fixed point argument in suitable Sobolev spaces, the global theorems of existence and uniqueness for the solution of the inverse problem are obtained.

Key words: nonlocal problem; the Gerasimov-Caputo derivative; integro-differential equation; inverse problem; Sobolev spaces; iteration method.

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1 Introduction and formulation of the main results

Let Ω be a bounded domain in $\mathbf{R}^d$ with sufficiently smooth boundary $\partial\Omega$ and $\Sigma_0^T = (0,T)\times\partial\Omega$. In this paper, we will consider the following time-fractional integrodifferential diffusion equation in $Q_0^T := \{(t,x) : 0 < t < T,\, x \in \Omega\}$:

$$\left(\partial_t^\alpha u\right)(t,x) + Au(t,x) = \int_0^t k(t-s)u(s,x)ds + f(t,x), \tag{1.1}$$

with the Gerasimov–Caputo time fractional derivative ∂_t^α of order $0<\alpha<1$, defined by (see [1]):

$$\left(\partial_t^\alpha y\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} y'(\tau)d\tau, \quad y \in W^{1,1}(0,T),$$

where $\Gamma(\cdot)$ is the Gamma function and the operator A is a symmetric uniformly elliptic operator defined on $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$ given by

$$Au(t,x) = -\sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\sum_{j=1}^d a_{ij}(x)\frac{\partial}{\partial x_j}u(t,x)\right) + c(x)u(t,x), \quad (t,x)\in Q_0^T,$$

in which the coefficients satisfy

$$a_{ij} = a_{ji}, \quad 1 \le i,j \le d, \quad a_{ij}\in C^1(\bar\Omega), \quad c(x)\in C(\bar\Omega),\ c(x)\ge 0, \quad x\in\bar\Omega$$

and there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\bar{\xi}_j \geq \mu \sum_{i=1}^{d} |\xi_i|^2, \quad \text{for all} \quad x \in \bar{\Omega}, \quad \xi \in \mathbf{R}^d,$$

We supplement the equation (1.1) with the non-local initial condition

$$u(T,x) - \beta u(0,x) = \varphi(x), \quad x \in \Omega, \tag{1.2}$$

the boundary condition

$$u(t,x) = 0, \quad (t,x) \in \Sigma_0^T. \tag{1.3}$$

The time-fractional integro-differential equation presents an important model to simulate the effects of the "memory" on the system. The model is based partial differential equation, combining the fractional differentiation and integral term containing the unknown kernel function that leads to a time-fractional integrodifferential equation. In general, the model's nonlocal effects on the memory of history cannot be described by partial differential equations. Therefore, more and more researchers have studied the solution of integro-differential equations. Recently, because of the many applications, fractional integro-differential equations are the focus of a large body of results. The classical integer order derivative is a local operator, which is not adequate for a description of many processes in physics, mechanics, industrial finance, etc. The fractional derivative is a nonlocal operator, which is often used to model phenomena in heat mass transfer, medicine, viscoelastic materials, porous media, mathematical biology, and in particular the honeybee population, mathematical finance and economics and atmosphere pollution, beam vibration (see [2]-[4] references therein). For the integrodifferential equation with integer order $\alpha = 1$, an efficient strategy based on analytic semigroup theory was given to prove the existence and uniqueness of inverse memory kernel problems [5]-[18].

Furthermore, if $\beta = 0$, this problem is called the backward problem. The backward problem when $k(t) = 0$ was extensively studied by many authors (see, e.g., [19]-[21]), therefore we omit in that case. When $\beta \neq 0$, then the general form of this problem is well-learned by [22] and we solved the considered problem inspired by these articles. In practical situations, the function represents some physical property, which is very hard to measure directly in advance. Therefore, we consider an inverse problem of determining convolution memory function k from additional measurement on u with for all-natural condition β.

The inverse problem in this paper is to reconstruct $k(t)$ according to the additional data

$$\mathcal{L}[u(t,\cdot)] := \int_{\Omega} \phi(x)u(t,x)dx = h(t), \quad t \in [0,T] \tag{1.4}$$

where $\phi(x), h(t)$ are given functions. Here $h(t)$ is the measurement data representing the average temperature on a small part of Ω because the weight function ϕ is usually chosen to satisfy $\mathrm{supp}(\phi) \subset\subset \Omega$ in applications.

For the reader's convenience, we present here the definitions from functional analysis and fractional calculus theory.

For integers m, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ (see [26]) and $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of space $H^m(\Omega)$. For a given Banach space V on Ω, we use the notation $C([0,T];V)$ to denote the following space:

$$C([0,T];V) := \left\{u : \|u(t)\|_V \text{ is continuous in } t \text{ on } [0,T]\right\}.$$

We endow $C([0,T];V)$ with the following norm making it to be a Banach space:

$$\|u\|_{C([0,T];V)} = \max_{0\le t\le T} \|u(t)\|_V.$$

The space

$$L^p(0,T;V)$$

consists of all strongly measurable functions $u : (0,T) \to V$ with

$$\|u\|_{L^p(0,T;V)} := \left(\int_0^T \|u(t)\|_V^p dt\right)^{1/p} < \infty$$

for $1 \le p < \infty$. In addition, we define Banach space X_0^T by

$$X_0^T := \left\{u : u \in C([0,T];D(A^\gamma)) \text{ and } \partial_t u \in L^1(0,T;L^2(\Omega))\right\},$$

where $\partial_t u$ means a distributional sense. Furthermore, we set the topological product

$$Y_0^T = X_0^T \times L^1(0,T)$$

endowed with the norm

$$\|(u,k)\|_{Y_0^T} := \|u\|_{C([0,T];D(A^\gamma))} + \|u_t\|_{L^1(0,T;L^2(\Omega))} + \|k\|_{L^1(0,T)}.$$

In this article, we consider the following inverse problem:

Inverse problem. Find $(u,k) \in X_0^T \times L^1(0,T)$ to satisfy (1.1)-(1.3) and the additional measurement (1.4).

The nonlinear problems are more difficult at the same time, it is both interesting and practical. Therefore more articles are focused on linear inverse problems, for example finding the right-hand side or data functions (initial, boundary) are well learned by many authors, especially [20]-[25]. However, as we said the nonlinear inverse problems are not much considered. A nonlinear inverse problem for a fractional diffusion equation is considered in [27]-[18], and they proved local existence, global uniqueness, and stability. Except for these works, we study the problem of determining the memory function from the integro-differential equation. Furthermore, we applied a new method to solve nonlinear inverse problem which is introduced in [37]-[40].

It is well-known that the operator A has only real and simple eigenvalues λ_n, and with suitable numbering, we have $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lim_{k\to\infty} \lambda_n = \infty$. By e_n, we denote the eigenfunction corresponding to λ_n, which satisfies $\|e_n\|^2_{L^2(\Omega)} = (e_n, e_n) = 1$, where $(\cdot,\cdot)$ denotes the inner product in Hilbert space $L^2(\Omega)$ and λ_n, e_n satisfy $Ae_n = \lambda_n e_n$, $e_n(x) = 0$, $x \in \partial\Omega$, $\{e_n\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ is an orthonormal basis of $L^2(\Omega)$.

Now we define the fractional power operator A^γ for $\gamma \in \mathbf{R}$ (e.g. [30]). Then for $\gamma \in \mathbf{R}$, we define a Hilbert space $D(A^\gamma)$ by

$$D(A^\gamma) := \Big\{u \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma}|(u, e_n)|^2 < \infty\Big\}, \quad A^\gamma u = \sum_{n=1}^{\infty} \lambda_n^{\gamma}(u, e_n)e_n$$

with the norm

$$\|u\|_{D(A^\gamma)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma}|(u, e_n)|^2\right)^{1/2}.$$

We note that the norm $\|u\|_{D(A^\gamma)}$ is stronger than $\|u\|_{L^2(\Omega)}$ for $\gamma > 0$. Since $D(A^\gamma) \subset L^2(\Omega)$, identifying the dual $(L^2(\Omega))'$ which itself, we have $D(A^\gamma) \subset L^2(\Omega) \subset (D(A^\gamma))'$. We set $D(A^{-\gamma}) = (D(A^\gamma))'$, which consists of bounded linear functional on $D(A^\gamma)$. For $u \in D(A^{-\gamma})$ and $\varphi \in D(A^\gamma)$, the value obtained by operating u to φ is denoted by ${}_{-\gamma}\langle \cdot, \cdot\rangle_\gamma$. $D(A^{-\gamma})$ is a Hilbert space with the norm:

$$\|\varphi\|_{D(A^{-\gamma})} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\gamma}|{}_{-\gamma}\langle u, e_n\rangle_\gamma|^2\right)^{\frac{1}{2}}.$$

We further note that

$${}_{-\gamma}\langle u, \varphi\rangle_\gamma = (u, \varphi) \quad \text{if } u \in L^2(\Omega) \text{ and } \varphi \in D(A^\gamma)$$

(see e.g., [20], Chapter V in [31]).

Furthermore, we introduce the Mittag-Leffler function, as presented in [32]:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z \in \mathbf{C}$$

with $\mathrm{Re}(\rho) > 0$ and $\mu \in \mathbf{C}$. It is known that $E_{\rho,\mu}(z)$ is an entire function in $z \in \mathbf{C}$. If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

In what follows we need the asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument. The well-known estimate has the form (see, e.g., [32], p.136)

$$|E_{\rho,\mu}(-t)| \le \frac{C}{1+t}, \quad t > 0. \tag{1.5}$$

This estimate essentially follows from the following asymptotic estimate (see, e.g., [32], p.134)

$$E_{\rho,\mu}(-t) = \frac{t^{-1}}{\Gamma(\mu - \rho)} + O(t^{-2}). \tag{1.6}$$

We will also use a coarser estimate with positive number λ and $0 < \varepsilon < 1$:

$$\left|t^{\rho-1}E_{\rho,\rho}(-\lambda t^\rho)\right| \le \frac{Ct^{\rho-1}}{1+(\lambda t^\rho)^2} \le C\lambda^{\varepsilon-1}t^{\varepsilon\rho-1}, \quad t > 0, \tag{1.7}$$

which is easy to verify (see, [22]).

Proposition 1.1. *The Mittag-Leffler function of negative argument $E_{\alpha,1}(-x)$ is completely monotonic (c.m.) for all $0 \le \alpha \le 1$ and*

$$0 < E_{\alpha,1}(-x) < 1.$$

Proposition 1.2. *(see [20]) For $\lambda > 0$, $\alpha > 0$ and positive integer $m \in \mathbf{N}$, we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0$$

and

$$\frac{d}{dt}\left(tE_{\alpha,2}(-\lambda t^\alpha)\right) = E_{\alpha,1}(-\lambda t^\alpha), \quad \partial_t^\alpha (E_{\alpha,1}(-\lambda t^\alpha)) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t \ge 0.$$

Throughout this paper, we set $0 < \varepsilon < 1$ and $\gamma > 0$ such that

$$\gamma_0 \ge \gamma > \begin{cases} \max\left\{\varepsilon, \frac{d}{4} - 1, \gamma_0 + \varepsilon - 1\right\}, & d \ge 4, \\ \max\left\{\varepsilon, \gamma_0 + \varepsilon - 1\right\}, & d = 1, 2, 3. \end{cases}$$

We make the following assumptions:
(C1) $\varphi \in D(A^{\gamma_0})$, $f \in X_0^T$;
(C2) $h(T) - \beta h(0) = \mathcal{L}[\varphi]$, $\mathcal{L}[Au](0) = \mathcal{L}[f](0)$;
(C3) $\partial_t^\alpha h \in C^1[0,T]$ and $\partial_t^\alpha h(0) = 0$ and satisfy the condition $h(0) \neq 0$;
(C4) $\phi \in H_0^2(\Omega)$.

Remark 1. *(C2) is the consistency condition for our problem (1.1)-(1.4), which guarantees that the inverse problem (1.1)-(1.4) is equivalent to (2.22) and (2.23) (see Lemma 2.4).*

Remark 2. *In (C3) implies $h \in C^1[0,T]$, which will be used in Lemma 2.4 and Lemma 2.5. Indeed, by $\partial_t^\alpha h(t) = D_t^{-(1-\alpha)} h(t)$ and $D_t^{1-\alpha} D_t^{-(1-\alpha)} h'(t) = h'(t)$ (see [35], (2.1.31)), we have*

$$h'(t) = D_t^{1-\alpha} D_t^{-(1-\alpha)} h'(t) = D_t^{1-\alpha} \partial_t^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-1} \partial_t^\alpha h(s) ds$$

$$= -\frac{1}{\Gamma(1+\alpha)} \frac{d}{dt} \int_0^t \partial_t^\alpha h(s) d(t-s)^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\partial_t^\alpha h)'(s) ds, \tag{1.10}$$

where D_t^α is the Riemann-Liouville fractional derivative, defined by

$$D_t^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds$$

for $0 < \alpha < 1$. Clearly, $\int_0^t (t-s)^{\alpha-1} ds \le C(\alpha, T)$, due to $-1 < \alpha - 1 < 0$. Therefore, from $\partial_t^\alpha h \in C^1[0,T]$ and (1.10), we conclude that $h \in C^1[0,T]$.

Furthermore, the condition (C3) is not empty. Indeed, let $h(t) = t^\nu + C$, where $\nu \geq \alpha + 1$ and $C = const \neq 0$. Then, we have

$$\partial_t^\alpha((t^\nu + C))(t) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha}.$$

Remark 3. *In engineering, $\phi(x)$ can be thought of as an internal (tiny) sensor (see [33], [34]) measuring the mean temperature in the measurement area. Hence supp(ϕ) is always chosen small enough to make the measurement area very small. Therefore hypothesis (C4) is reasonable.*

Remark 4. *Its well known, if $y \in W^{1,1}(0,T)$, then y absolutely continuous function. Hence, the fractional derivative $\partial_t^\alpha y$ exist almost everywhere on $[0,T]$ (see, [35], p. 92). Furthermore, a continuous function $y : [0,T] \to \mathbf{R}$ is absolutely continuous if and only its distributional derivative is an $L^1(0,T)$ function (see, Theorem 1 in Section 4.9, [36]). Therefore, the functional class of the solution to the problem (1.1)-(1.3) is well defined.*

Our main result is as follows:

Theorem. *Under hypotheses (C1)-(C5), there exists a unique solution $(u,k) \in Y_0^T$ of the inverse problem (1.1)-(1.4) for any $T > 0$.*

Note, that the (C5) condition is given below (see, p. 7).

The plan of the paper:

• Section 2 contains the preliminary lemmas that will be necessary to prove our main result.

• In section 3 is devoted to proving the existence local in time of the inverse problem (1.1)-(1.4) using the Banach principle and global uniqueness of our inverse problem.

• Finally, in Section 4, we prove our main result by fixed point argument, thanks to the preliminary results of Sections 2 and 3.

2 Analysis of the backward problem and prior estimates

In this section, we will study the backward problem (1.1)-(1.3) for different values of $\beta \neq 0$. For convenience, let $F = k * u + f$. Then, we rewrite the backward problem as follows

$$\begin{cases} \partial_t^\alpha v(t,x) + Av(t,x) = F(t,x), & (t,x) \in Q_0^T, \\ v(T,x) - \beta v(0,x) = \varphi(x), & x \in \Omega, \\ v(t,x) = 0, & (t,x) \in \Sigma_0^T, \end{cases} \tag{2.1}$$

Before we present and prove our results, let us dwell on the existing literature. As we said, if $\beta \neq 0$, this problem is well learned in [22]. But, our main task is to study the nonlinear inverse problem (1.1)-(1.4). Therefore, here we present some cases which are given in [22].

To solve the problem (2.1), we divide it into two auxiliary problems:

$$\begin{cases} \partial_t^\alpha w(x,t) + Aw(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ w(x,0) = 0, & x \in \Omega, \\ w(x,t) = 0, & (x,t) \in \Sigma_0^T, \end{cases} \tag{2.1a}$$

and

$$\begin{cases} \partial_t^\alpha \omega(x,t) + A\omega(x,t) = 0, & (x,t) \in Q_0^T, \\ \omega(x,T) - \beta\omega(x,0) = \psi(x), & x \in \Omega, \\ \omega(x,t) = 0, & (x,t) \in \Sigma_0^T. \end{cases} \tag{2.1b}$$

Problem (2.1b) is a particular case of problem (2.1). If $\psi = \varphi - w(x,T)$ and $w(x,t)$ and $\omega(x,t)$ are the corresponding solutions, then it is easy to verify that the function $v(t,x) = w(t,x) + \omega(t,x)$ is a solution the to problem (2.1). Therefore, it is enough to solve the auxiliary problems.

Let us denote

$$Z_\beta(t)a(x) = \sum_{n=1}^{\infty} \rho_n(T)(a,e_n)E_{\alpha,1}(-\lambda_n t^\alpha)e_n(x), \quad (x,t) \in Q_0^T$$

for $a \in L^2(\Omega)$, where

$$\rho_n(T) := \frac{1}{E_{\alpha,1}(-\lambda_n T^\alpha) - \beta}. \tag{2.2}$$

Let us first get acquainted with the analysis of function (2.2) given in [22].

1st *case:* $\beta = 0$. Then, $E_{\alpha,1}(-\lambda_n T^\alpha) \neq 0$, but the Mittag-Leffler function can asymptotically tend towards zero (see (1.6)). Therefore, in this case we have:

$$|\rho_n| \leq C_\beta \lambda_n T^\alpha.$$

2nd *case:* $0 < \beta < 1$. Then, in view of Proposition 1.1, there is a unique $\lambda_0 > 0$ such that $E_{\alpha,1}(-\lambda_0 T^\alpha) = \beta$. If $\lambda_n \neq \lambda_0$ for all $n \in \mathbf{N}$, then

$$|\rho_n| \leq C_\beta$$

the estimate is held with some constant $C_\beta > 0$. Therefore, if $\beta \notin (0,1)$ or $\beta \in (0,1)$, but $\lambda_n \neq \lambda_0$ for all $n \in \mathbf{N}$, then the formal solution of problem (2.1b) has the form

$$\omega(t,x) = Z_\beta \psi(x), \qquad (t,x) \in Q_0^T. \tag{2.3}$$

3rd *case:* $0 < \beta < 1$ and $\lambda_n = \lambda_0$ for $n = n_0, n_0+1, ..., n_0+p_0-1$ where p_0 is the multiplicity of the eigenvalue λ_{n_0}. Then the non-local problem (2.1b) has a solution if the boundary function $\psi(x)$ satisfies the following orthogonality conditions

$$\psi_n = (\psi, e_n) = 0, \quad n \in \mathcal{K}, \quad \mathcal{K} = \{n_0, n_0+1, ..., n_0+p_0-1\}. \tag{2.4}$$

For all other n we have

$$|\rho_n| \leq C_\beta, \quad n \notin \mathcal{K}. \tag{2.5}$$

Thus, the formal solution to the problem (2.1b) in this case has the form

$$\omega(t,x) = \sum_{n\notin\mathcal{K}} \rho_n(T)(\psi, e_n)E_{\alpha,1}(-\lambda_n t^\alpha)e_n(x) + \sum_{n\in\mathcal{K}} \omega_n(0)E_{\alpha,1}(-\lambda_n t^\alpha)e_n(x). \qquad (2.6)$$

Let H be a separable Hilbert space. By combining the above three cases the authors [22] obtained the following result:

Lemma 2.1. *Let $\psi \in H$.*

(i) If $\beta \notin (0,1)$ or $\beta \in (0,1)$, but $\lambda_n \neq \lambda_0$ for all $n \in \mathbf{N}$, then problem (2.1b) has a unique solution and this solution has the form (2.3);

(ii) If $\beta \in (0,1)$ and $\lambda_n = \lambda_0$, $n \in \mathcal{K}$, then we assume that the orthogonality conditions (2.4) are satisfied. The solution of problem (2.1b) has the form (2.6) with arbitrary coefficients $\omega_n(0)$, $n \in \mathcal{K}$.

Now, we turn to study the problem (2.1a). First, we define a weak solution to (2.1a).

Definition 1. *We call $u(t,x)$ a weak solution to (2.1a), if equation holds in $L^2(\Omega)$ and $w(\cdot,t) \in D(A^{\frac{1}{2}})$ for almost all $t \in (0,T)$ and $w \in C([0,T];D(A^{-\gamma}))$,*

$$\lim_{t\to 0} \|w(t,\cdot)\|_{D(A^{-\gamma})} = 0$$

with some $\gamma > 0$.

A form problem (2.1a) was also studied in [22], but in our case, the smoothness increased even more.

Lemma 2.2. *Let $F \in C([0,T];D(A^\gamma))$. Then there exists a unique weak solution $w \in C([0,T];L^2(\Omega))$ to (2.1a) such that $\partial_t^\alpha w \in C([0,T];L^2(\Omega))$. In particular, for any satisfying $\gamma > \frac{d}{4} - 1$, we have $w \in C([0,T];D(A^{-\gamma}))$,*

$$\lim_{t\to 0} \|w(t,\cdot)\|_{D(A^{-\gamma})} = 0,$$

and if $d = 1,2,3$, then

$$\lim_{t\to 0} \|w(t,\cdot)\|_{L^2(\Omega)} = 0.$$

Moreover, there exists a constant $c > 0$ such that

$$\|w\|_{C([0,T];D(A^\gamma))} + \|\partial_t^\alpha w\|_{C([0,T];L^2(\Omega))} \leq c\|F\|_{C([0,T];D(A^\gamma))} T^{\alpha\varepsilon} \qquad (2.7)$$

and we have

$$w(t,x) = \int_0^t A^{-1}Z'(t-s)F(s,x)ds, \qquad (t,x) \in Q_0^T, \qquad (2.8)$$

using of

$$Z'(t)a(x) = \sum_{n=1}^{\infty} \lambda_n (a,e_n) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)e_n(x),$$

the space in (2.7).

Proof
Let $w_n(t) = (w(\cdot,t), e_n)$, $n \geq 1$. Assume that problem (2.1a) has a unique solution w given by

$$w(t,x) = \sum_{n=1}^{\infty} w_n(t) e_n(x),$$

where $w_n, (n = 1, 2, ...)$ are solutions of Cauchy problems:

$$\begin{cases} \partial_t^\alpha w_n(t) + \lambda_n w_n(t) = F_n(t), \quad 0 < t < T, \\ w_n(0) = 0, \end{cases} \tag{2.9}$$

here $F_n(t) = (F(t,\cdot), e_n)$ and λ_n are the eigenvalues corresponding to that eigenfunctions e_n. According to Theorem 5.15 in [35] (see p. 323), there exists a unique solution to (2.9) and it is defined as

$$w_n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F_n(s) ds.$$

Thus, we can formally obtain a formula (2.8).

Because of the Parseval equality and of the generalized Minkowski inequality, and (1.7), we have

$$\|w(t,\cdot)\|^2_{D(A^\gamma)} = \sum_{n=1}^{\infty} \lambda_n^{2\gamma} \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) F_n(s) ds \right|^2$$

$$\leq C^2 \left[\int_0^t (t-s)^{\alpha\varepsilon-1} \lambda_1^{\varepsilon-1} \Big(\sum_{n=1}^{\infty} |\lambda_n^\gamma F_n(s)|^2 \Big)^{1/2} ds \right]^2 \leq C^2(\alpha,\varepsilon,\lambda_1) t^{2\alpha\varepsilon} \max_{0\leq s\leq t} \|F(s)\|^2_{D(A^\gamma)}$$

for all $t \in [0,T]$. Due to the $0 < \varepsilon < 1$, then we have

$$\|w(t,\cdot)\|^2_{D(A^\gamma)} \leq C^2 \|F\|^2_{C([0,t];D(A^\gamma))} t^{2\alpha\varepsilon}. \tag{2.9}$$

Furthermore, given the condition of Lemma 2.2, for $F \in C([0,T]; D(A^\gamma))$ and as (2.9), we have

$$\|Aw(t,\cdot)\|^2_{L^2(\Omega)} \leq C^2(\alpha,\varepsilon,\lambda_1) t^{2\alpha\varepsilon} \|F\|^2_{C([0,t];D(A^\gamma))}. \tag{2.10}$$

So, we combine the above result with Eq. (2.1a), we have $\partial_t^\alpha w(t,x) \in C([0,T]; L^2(\Omega))$ and

$$\|\partial_t^\alpha w(t,\cdot)\|^2_{L^2(\Omega)} \leq C^2 \|F\|^2_{C([0,t];D(A^\gamma))} t^{2\alpha\varepsilon}, \quad t > 0. \tag{2.11}$$

Summing up (2.9) and (2.11) yields (2.7). Finally, we have to prove $\lim\limits_{t\to 0} \|w(t,\cdot)\|_{D(A^{-\gamma})} = 0$. This implies from Theorem 2.2 (i) in [20] and

$$\text{ess sup} f = \max f$$

for sense a continuous functions f on $[a,b]$.

The uniqueness of the solution can be proved by the standard technique based on the completeness of the set of eigenfunctions $\{e_n\}$ in $L^2(\Omega)$ and so, in $D(A^\gamma)$ too (by the definition of $D(A^\gamma)$). Lemma 2.2 is fully proven.

Further in this article, we consider (1.1)-(1.3) for the case when

$$\beta \notin [0,1], \quad \text{or} \quad \beta \in (0,1), \text{ but } \lambda_n \neq \lambda_0, \text{ for all } n \in \mathbf{N}. \tag{C5}$$

Remark 5. *The problem (2.1) has a solution even if β equals one. However, when obtaining an equivalent problem to the inverse problem (1.1)-(1.4), it is necessary that β is not equal to 1. That is why we consider $\beta \notin [0,1]$ (see Lemma 2.4).*

Combining the functions (2.3) and (2.8), we get

$$v(t,x) = \sum_{n=1}^{\infty} \Big(\frac{\varphi_n - w_n(T)}{E_{\alpha,1}(-\lambda_n T^\alpha) - \beta} E_{\alpha,1}(-\lambda_n t^\alpha) + w_n(t) \Big) e_n(x), \tag{2.12}$$

where

$$w_n(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha)(F(s,\cdot), e_n) ds. \tag{2.13}$$

The following lemma is the result of the regularity of the solution v of the problem (2.1).

Lemma 2.3. *Let $\varphi \in D(A^{\gamma_0})$, $F \in X_0^T$. Then, the unique solution $v \in X_0^T$ to the problem (2.1) is represented by (2.12), such that*

$$\|v\|_{X_0^T} \le C\Big[(1+T^{\alpha\varepsilon})\|\varphi\|_{D(A^{\gamma_0})} + T^{\alpha\varepsilon}\|F\|_{C([0,T];D(A^\gamma)} + T^{\alpha\varepsilon}\|F(0,\cdot)\|_{D(A^\gamma)}$$

$$+T^{\alpha\varepsilon}\|\partial_t F\|_{L^1(0,T;L^2(\Omega))}\Big], \tag{2.14}$$

where C depends on β, α, λ_1.

Proof

According to Proposition 1.1 and (2.12), we have

$$\|v(t,\cdot)\|^2_{D(A^\gamma)} = \sum_{n=1}^{\infty} \lambda_n^{2\gamma} \Big| \frac{\varphi_n - w_n(T)}{E_{\alpha,1}(-\lambda_n T^\alpha) - \beta} E_{\alpha,1}(-\lambda_n t^\alpha) \Big|^2 + \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |w_n(t)|^2$$

$$\le C_\beta \lambda_1^{-2(\gamma_0-\gamma)} \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0} |\varphi_n|^2 + C_\beta \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |w_n(T)|^2 + \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |w_n(t)|^2 \tag{2.15}$$

due to $\gamma \le \gamma_0$. By (2.9), the third term of the last sum is estimated as follows:

$$\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |w_n(t)|^2 \le C^2(\alpha, \varepsilon, \lambda_1) t^{2\alpha\varepsilon} \|F\|^2_{C([0,t];D(A^\gamma))}. \tag{2.16}$$

Combining the (2.15) and (2.16), we have

$$\|v(t,\cdot)\|^2_{D(A^\gamma)} \le C^2(\beta, \gamma_1) \|\varphi\|^2_{D(A^{\gamma_0})}$$

$$+C^2(\alpha,\varepsilon,\lambda_1) T^{2\alpha\varepsilon} \|F\|^2_{C([0,T];D(A^\gamma))} + C^2(\alpha,\varepsilon,\lambda_1) t^{2\alpha\varepsilon} \|F\|^2_{C([0,t];D(A^\gamma))}, \quad t \in [0,T].$$

On the other hand, using Proposition 1.1, we have

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$$

from which it follows that

$$
\begin{aligned}
v_t(t,x) = & -\sum_{n=1}^{\infty} \lambda_n \frac{\varphi_n - w_n(T)}{E_{\alpha,1}(-\lambda_n T^{\alpha}) - \beta} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) e_n(x) \\
& + \sum_{n=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) F_n(0) e_n(x) \\
& + \sum_{n=1}^{\infty} \left(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^{\alpha}) F_n'(s) ds \right) e_n(x) \\
& := \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned} \tag{2.17}
$$

For I_1, by (1.7) and (C5), we have

$$
\begin{aligned}
\|\mathrm{I}_1(t,\cdot)\|_{L^2(\Omega)}^2 & \le \sum_{n=1}^{\infty} \left| \lambda_n \frac{\varphi_n}{E_{\alpha,1}(-\lambda_n T^{\alpha}) - \beta} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) e_n(x) \right|^2 \\
& \le C_\beta^2 t^{2\alpha\varepsilon-2} \sum_{n=1}^{\infty} \lambda_n^{2\varepsilon} |\varphi_n|^2 \le C_\beta^2 t^{2\alpha\varepsilon-2} \|\varphi\|_{D(A^{\varepsilon})}^2.
\end{aligned}
$$

So, by $D(A^{\gamma_0}) \subset D(A^{\varepsilon})$ for $\varepsilon < \gamma_0$, we have

$$
\|\mathrm{I}_1(t,\cdot)\|_{L^2(\Omega)} \le C_\beta \lambda_1^{-\gamma_0+\varepsilon} t^{\alpha\varepsilon-1} \|\varphi\|_{D(A^{\gamma_0})},
$$

or

$$
\|\mathrm{I}_1\|_{L^1(0,T;L^2(\Omega))} \le \frac{C_\beta}{\alpha\varepsilon} \lambda_1^{-\gamma_0+\varepsilon} T^{\alpha\varepsilon} \|\varphi\|_{D(A^{\gamma_0})}. \tag{2.18}
$$

Now we estimate I_2. Again using Proposition 1.1 and estimate (1.7), we obtain

$$
\begin{aligned}
\|\mathrm{I}_2(t,\cdot)\|_{L^2(\Omega)}^2 & \le \sum_{n=1}^{\infty} \left| \lambda_n \frac{w_n(T)}{E_{\alpha,1}(-\lambda_n T^{\alpha}) - \beta} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) e_n(x) \right|^2 \\
& \le C_\beta^2 \sum_{n=1}^{\infty} \left| \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-s)^{\alpha}) F_n(s) ds \right|^2 \\
& \le C_\beta^2 t^{2\alpha\varepsilon-2} \left[\int_0^T \left(\sum_{n=1}^{\infty} \left(\lambda_n^{\gamma} F_n(s) \lambda_n^{-(\gamma-2\varepsilon-1)} \right)^2 \right)^{\frac{1}{2}} (T-s)^{\alpha\varepsilon-1} ds \right]^2.
\end{aligned}
$$

So, in view of $2\varepsilon - 1 < \varepsilon < \gamma$ for $0 < \varepsilon < 1$, yields $\lambda_n^{-\gamma+2\varepsilon-1} \le \lambda_1^{-\gamma+2\varepsilon-1}$, and we have

$$
\|\mathrm{I}_2(t,\cdot)\|_{L^2(\Omega)} \le C(\beta,\alpha,\varepsilon) \lambda_1^{-\gamma+\varepsilon-1} t^{\alpha\varepsilon-1} T^{\alpha\varepsilon} \|F\|_{C([0,T];D(A^{\gamma})},
$$

or

$$
\|\mathrm{I}_2\|_{L^1(0,T;L^2(\Omega))} \le C(\beta,\alpha,\varepsilon,\lambda_1) T^{2\alpha\varepsilon} \|F\|_{C([0,T];D(A^{\gamma})}. \tag{2.19}
$$

Besides, for I_3, we have

$$
\|\mathrm{I}_3(t,\cdot)\|_{L^2(\Omega)}^2 \le \sum_{n=1}^{\infty} \left| t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) F_n(0) e_n(x) \right|^2 \le C^2 t^{2\alpha\varepsilon-2} \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |F_n(0)|^2 \lambda_n^{-2\gamma+2\varepsilon-2}
$$

$$\leq C^2\lambda_1^{-2\gamma+2\varepsilon-2}t^{2\alpha\varepsilon-2}\|F(0,\cdot)\|^2_{D(A^\gamma)}.$$

So, we get

$$\|\mathrm{I}_3\|_{L^1(0,T;L^2(\Omega))} \leq \frac{C\lambda_1^{-\gamma+\varepsilon-1}}{\alpha\varepsilon}T^{\alpha\varepsilon}\|F(0,\cdot)\|_{D(A^\gamma)}. \tag{2.20}$$

Similarly, by (2.17) and Young inequality for the convolution, we have the following estimate for I_4:

$$\|\mathrm{I}_4\|_{L^1(0,T;L^2(\Omega))} \leq \frac{C(\alpha)}{\alpha\varepsilon}\lambda_1^{\varepsilon-1}T^{\alpha\varepsilon}\|\partial_t F\|_{L^1(0,T;L^2(\Omega))}. \tag{2.21}$$

This completes the proof of this lemma.

To study the main problem (1.1)-(1.4), we consider the following auxiliary inverse non-local initial and boundary value problem.

Lemma 2.4. *Let (C1)-(C5) be held. Then the problem of finding a solution of (1.1)-(1.4) is equivalent to the problem of determining the functions $u(t,x) \in X_0^T$ and $l(t) \in AC[0,T]$ satisfying*

$$\begin{cases} \left(\partial_t^\alpha u\right)(t,x) + Au(t,x) = \int_0^t k(t-s)u(s,x)ds + f(t,x), \quad (t,x) \in Q_0^T, \\ u(T,x) - \beta u(0,x) = \varphi(x), \quad x \in \Omega \\ u(t,x) = 0, \quad (t,x) \in \Sigma_0^T \end{cases} \tag{2.22}$$

and

$$k(t) = \frac{1}{h(0)}D_t\mathcal{N}[u,l](t), \qquad 0 \leq t \leq T, \tag{2.23}$$

where $D_t := (d/dt)$, $\mathcal{N}$ is defined by (2.25) below and

$$l(t) = \int_0^t k(\tau)d\tau.$$

On the other hand, if (2.22)-(2.23) has a solution and the technical condition (C1)-(C5) holds, then there exists a solution to the inverse problem (1.1)-(1.4).

Remark 6. *From Lemma 2.4, we know that (2.22)-(2.23) is an equivalent form of the original inverse problem (1.1)-(1.4). So, in the next sections, we discuss (2.22)-(2.23), other than the original one.*

Proof

The solution $(u(t,x),k(t)) \in Y_0^T$ of our inverse problem (1.1)-(1.4) is also a solution to the problem (2.22) in Y_0^T. Because the problem (2.22) is the same as (1.1)-(1.3), therefore, we should show only (2.23). Let the two $\{u(t,x),k(t)\}$ functions be a solution of problem (1.1)-(1.4). Taking into account Remark 1.2, and applying $\mathcal{L}$ to both sides of (1.1) yields

$$\partial_t^\alpha h(t) + \mathcal{L}[Au](t) = \int_0^t k(t-\tau)h(\tau)d\tau + \mathcal{L}[f](t). \tag{2.24}$$

We note that $l(t) = \int_0^t k(\tau)d\tau$. Then by integration by parts, we get the following equality:

$$\int_0^t k(\tau)h(t-\tau)d\tau = h(0)l(t) + \int_0^t l(t-\tau)h'(\tau)d\tau. \tag{2.25}$$

With the help of (2.25), we can rewrite (2.24) as

$$\partial_t^\alpha h(t) + \mathcal{L}[Au](t) = h(0)l(t) + \int_0^t l(t-\tau)h'(\tau)d\tau + \mathcal{L}[f](t). \tag{2.26}$$

Due to (C3), we can solve this equation by $l(t)$ and we have

$$\begin{aligned} h(0)l(t) &= \partial_t^\alpha h(t) + \mathcal{L}[Au](t) - \int_0^t l(t-\tau)h'(\tau)d\tau - \mathcal{L}[f](t) \\ &:= \mathcal{N}[u,l]. \end{aligned} \tag{2.27}$$

Furthermore, by differentiating (2.27) concerning t, we get (2.23).

Now we assume that (u,k) satisfies (2.22)-(2.23). In order to prove that $\{u,k\}$ is the solution to the inverse problem (1.1)-(1.4), it suffices to show that $\{u,k\}$ satisfies (1.4).

Applying $\mathcal{L}$ to the Eq. in (2.22), we have

$$\partial_t^\alpha \mathcal{L}[u](t) + \mathcal{L}[Au](t) = \int_0^t k(t-s)\mathcal{L}[u](s)ds + \mathcal{L}[f](t). \tag{2.28}$$

On the other hand, from (C2), we easily see that

$$\mathcal{N}[u,l](0) = 0.$$

We get (2.27) by integrating (2.23) over $[0,t]$. From (2.27), we conclude that

$$\begin{aligned} 0 &= -h(0)l(t) + \partial_t^\alpha h(t) + \mathcal{L}[Au](t) - \int_0^t l(t-s)h'(s)ds - \mathcal{L}[f](t) \\ &= \partial_t^\alpha h(t) + \mathcal{L}[Au](t) - \int_0^t k(t-s)h(s)ds - \mathcal{L}[f](t) \end{aligned}$$

or

$$\mathcal{L}[f](t) = \partial_t^\alpha h(t) + \mathcal{L}[Au](t) - \int_0^t k(t-s)h(s)ds. \tag{2.29}$$

Then substituting (2.29) into (2.28), and using (C3), we have that $P(t) := \mathcal{L}[u](t) - h(t)$ satisfy

$$\begin{cases} \partial_t^\alpha P(t) = \int_0^t k(t-s)P(s)ds, & t > 0, \\ P(T) - \beta P(0) = 0. \end{cases} \tag{2.30}$$

Let $k : (0,T) \to \mathbf{R}_+$ is integrable and condition (C5) (see Remark 5) be satisfied. Then, the boundary value problem for differential equation (2.30) with fractional order has a unique solution on $[0,T]$, certainly in $C^1[0,T]$ (see, [41]). Therefore, it is easy to see that the solution to the problem (2.30) is only trivial, that is, $P(t) \equiv 0$ for all $t \in [0,T]$, which implies $\mathcal{L}[u](t) - h(t) = 0$, $0 \le t \le T$, i.e., the condition (1.4) is satisfied. This completes the proof of Lemma 2.4.

Example 1. *Let $\beta = 2$. Consider the case $n = 1$ and $Q_0^T = (0,1)\times(0,T)$ with $T = \sqrt{2}$, and $A := -\partial_x^2$. We fix the initial condition φ, source term $f(t,x)$ and measurement data $\phi(x)$, $h(t)$ for any $x \in (0,1)$ and $t > 0$ as the following:*

$$\begin{cases} f(t,x) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + 2\left(\pi^2(1+t^2) - \frac{t^\alpha}{\alpha} - \frac{2\Gamma(\alpha)}{\Gamma(3+\alpha)} t^{2+\alpha}\right)\sin(\pi x), \\ \varphi(x) = 2\sin(\pi x), \\ \phi(x) = \sin(\pi x), \\ h(t) = t^2 + 1. \end{cases}$$

Then, the exact solution is $u(t,x) = 2(1+t^2)\sin(\pi x)$ and $k(t) = t^{\alpha-1}$. Note, that the all given data satisfy conditions (C1)-(C5).

At the end of this section, we give a technical lemma that will be used to estimate k in suitable Sobolev space in the next section.

Lemma 2.5. *Let (C1), (C3) and (C4) hold. Then for all $u \in X_0^T$ and all $k \in L^1(0,T)$, there exists a constant $C > 0$ depending on h, φ and f, but independent of T, such that*

$$\|D_t\mathcal{N}[u,l]\|_{L^1(0,T)} \le C\left(T + \|u_t\|_{L^1(0,T;L^2(\Omega))} + T\|k\|_{L^1(0,T)}\right) \tag{2.31}$$

where $\mathcal{N}$ is the same as that in (2.26).

Proof

By integration by parts, it follows from (C4) that

$$\mathcal{L}\{A[u_t]\}(t) = \int_\Omega \phi(x)A[u_t](t,x)dx = \int_\Omega A[\phi]u_t(t,x)dx, \tag{2.32}$$

which gives

$$\|\mathcal{L}\{A[u_t]\}\|_{L^1(0,T)} = \int_0^T \left|\int_\Omega A[\phi]u_t(t,x)dx\right| dt$$

$$\le \int_0^T \left(\int_\Omega |A[\phi]|^2 dx\right)^{1/2} \left(\int_\Omega |u_t|^2 dx\right)^{1/2} dt \le \|\phi\|_{H^2(\Omega)}\|u_t\|_{L^1(0,T;L^2(\Omega))}, \tag{2.33}$$

where we have used the Hölder's inequality.

Therefore, from (2.27) and the Young inequality for the convolution, we have

$$\|D_t\mathcal{N}[u,l]\|_{L^1(0,T)} \le \|(\partial_t^\alpha h)'\|_{L^1(0,T)} + \|\mathcal{L}\{A[u_t]\}\|_{L^1(0,T)}$$

$$+\|k * h'\|_{L^1(0,T)} + \|\mathcal{L}[f_t]\|_{L^1(0,T)} \le \|(\partial_t^\alpha h)'\|_{L^1(0,T)} + \|\phi\|_{H^2(\Omega)}\|u_t\|_{L^1(0,T;L^2(\Omega))}$$

$$+T\|k\|_{L^1(0,T)}\|h'\|_{C[0,T]} + \|\phi\|_{L^2(\Omega)}\|f_t\|_{L^1(0,T;L^2(\Omega))}. \tag{2.34}$$

From the last summation, we obtain the desired estimate (2.31). This completes the proof of Lemma 2.5.

3 Well-posedness of inverse problem

We can now prove the existence of a solution to our inverse problem, i.e. Theorem 1.1, which proceeds by a fixed point argument. First, we define the function set

$$B_{\rho,T} = \big\{(\bar{u},\bar{k}) \in Y_0^T : \ \bar{u}(T,x) - \beta\bar{u}(0,x) = \varphi(x), \ \bar{u}(t,x) = 0, \ (t,x) \in \Sigma_0^T,$$
$$\|\bar{u}\|_{X_0^T} + \|\bar{k}\|_{L^1(0,T)} \leq \rho\big\}.$$

Here ρ is a large constant depending on the initial data φ, measurement data h, and number β.

For given $(\bar{u},\bar{k}) \in B_{\rho,T}$, we consider

$$\begin{cases} \partial_t^\alpha u(t,x) + Au(t,x) = F(t,x), & (t,x) \in Q_0^T, \\ u(T,x) - \beta u(0,x) = \varphi(x), & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in \Sigma_0^T, \end{cases} \tag{3.1}$$

where

$$F(t,x) = \int_0^t \bar{k}(t-s)\bar{u}(s,x)ds + f(t,x),$$

and

$$k(t) = \frac{1}{h(0)} D_t \mathcal{N}[u,\bar{l}](t) \tag{3.2}$$

to generate (u,k), where $\bar{l}(t) = \int_0^t \bar{k}(\tau)d\tau$, $\mathcal{N}$ is the same as those in (2.27).

Remark 7. *As [40], usually, we use $\bar{u}$ on the right-hand side of (3.2) to generate k. But, if we do so, we can not choose suitable T and R to prove $\|k\|_{L^1(0,T)} \leq R$, because there is a lack of T in the terms including u_t in (2.31). In this situation, the fixed point argument can not be applied to our problem. So we take the solution v to (4.1) to generate k. This process in principle is similar to the Gauss-Siedel iteration (see [37]-[39]).*

Further, we have

$$\|\bar{k} * \bar{u}\|_{C([0,T];D(A^\gamma))} \leq \|\bar{k}\|_{L^1(0,T)} \|\bar{u}\|_{C([0,T];D(A^\gamma))}, \tag{3.3}$$

which implies

$$\|\bar{k} * \bar{u}\|_{C([0,T];D(A^\gamma))} \leq C(\rho)$$

Using this result together with $f \in C([0,T];D(A^\gamma))$, we have

$$\int_0^t \bar{k}(t-s)\bar{u}(s,x)ds + f(x,t) \in C([0,T];D(A^\gamma)).$$

Hence, Lemma 2.3 ensures that there exists a unique solution $u \in X_0^T$ to (3.1). Then (3.2) defines the function $k(t)$ in terms of u. Furthermore, by Lemma 2.5, we have

$$\|k\|_{L^1(0,T)} \leq C\left(T + \|u_t\|_{L^1(0,T;L^2(\Omega))} + T\|\bar{k}\|_{L^1(0,T)}\right). \tag{3.4}$$

This implies that $k \in L^1(0,T)$.

Thus the mapping

$$\mathcal{S} : B_{\rho,T} \to Y_0^T, \quad (\bar{u}, \bar{k}) \to (u, k)$$

given by (3.1) and (3.2) is well-defined.

The next lemma shows that $\mathcal{S}$ maps $B_{\rho,T}$ into itself for sufficiently small $T > 0$. More precisely we have the following result:

Lemma 3.1. *Let (C1)-(C5) be hold. For* $(\bar{u}, \bar{k})$, $(\bar{U}, \bar{K}) \in B_{r,T}$, *define*

$$(u, k) = \mathcal{S}(\bar{u}, \bar{k}), \quad (U, K) = \mathcal{S}(\bar{U}, \bar{K}).$$

Then for properly small $\tau > 0$, *we have*

$$\|(u,k)\|_{Y_0^T} \le \rho$$

and

$$\|(u - U, k - K)\|_{Y_0^T} \le \frac{1}{2}\|(\bar{u} - \bar{U}, \bar{k} - \bar{K})\|_{Y_0^T} \tag{3.5}$$

for all $T \in (0, \tau]$.

Throughout, we use C to denote a constant that depends on Ω, α, β, the initial data φ, the known functions f, ϕ and measurement data h, but independent of ρ and T.

Proof

First we prove that the operator $\mathcal{S}(B_{\rho,T}) \subset B_{\rho,T}$ for sufficiently small T and suitable larger ρ. To simplify the calculations, we restrict $T \in (0,1]$. From Lemma 2.3, (3.3), we have

$$\|u\|_{X_0^T} \le C\Big[(1 + T^{\alpha\varepsilon})\|\varphi\|_{D(A^{\gamma_0})} + T^{\alpha\varepsilon}\left(\|\bar{k} * \bar{u}\|_{C([0,T];D(A^\gamma))} + \|f\|_{C([0,T];D(A^\gamma))}\right)$$

$$+T^{\alpha\varepsilon}\|f(0,\cdot)\|_{D(A^\gamma)} + T^{\alpha\varepsilon}\|\bar{k}\|_{L^1(0,T)}\|\bar{u}(0,\cdot)\|_{L^2(\Omega)} + T^{\alpha\varepsilon}\|\bar{k} * \bar{u}_t\|_{L^1(0,T;L^2(\Omega))} + T^{\alpha\varepsilon}\|f_t\|_{L^1(0,T;L^2(\Omega))}\Big]$$

$$\le C\Big((1 + T^{\alpha\varepsilon})\|\varphi\|_{D(A^{\gamma_0})} + T^{\alpha\varepsilon}\left(\rho^2 + \|f\|_{C([0,T];D(A^\gamma))}\right)$$

$$+T^{\alpha\varepsilon}\|f\|_{C([0,T];D(A^\gamma))} + \lambda_1^{-\gamma}T^{\alpha\varepsilon}\|\bar{k}\|_{L^1(0,T)}\|\bar{u}\|_{C([0,T];D(A^\gamma))} + \rho^2 T^{\alpha\varepsilon} + T^{\alpha\varepsilon}\|f_t\|_{L^1(0,T;L^2(\Omega))}\Big)$$

$$\le C\Big(1 + T^{\alpha\varepsilon} + \rho^2 T^{\alpha\varepsilon}\Big). \tag{3.6}$$

On the other hand, by (3.4), we have

$$\|k\|_{L^1(0,T)} \le C\left(T + \|u_t\|_{L^1(0,T;L^2(\Omega))} + T\|\bar{k}\|_{L^1(0,T)}\right) \le C\left(T + \|u\|_{X_0^T} + T\rho\right)$$

$$\le C\Big(1 + T + T\rho + T^{\alpha\varepsilon} + T^{\alpha\varepsilon}\rho^2\Big). \tag{3.7}$$

Then, adding up (3.6) and (3.7) leads to

$$\|(u,k)\|_{Y_0^T} \le C\omega_1(T,\rho) + C, \tag{3.8}$$

where the function $\omega_1(T)$ is of the form

$$\omega_1(T) = T + T\rho + T^{\alpha\varepsilon} + T^{\alpha\varepsilon}\rho^2$$

and therefore satisfies $\lim_{T\to+0} \omega_1(T,\rho) = 0$. Now we take ρ, such that $\rho = 2C$ with the constant C in (3.8). Then there exists $\tau_1 > 0$ such that

$$\|(u,k)\|_{Y_0^T} \le \rho \tag{3.9}$$

for all $T \in (0,\tau_1]$. That is, $\mathcal{S}$ maps $B_{\rho,T}$ into itself for each fixed $T \in (0, \min\{1,\tau_1\}]$.

Next, we check the second condition of contractive mapping $\mathcal{S}$. Let $(u,k) = \mathcal{S}(\bar{u},\bar{k})$ and $(U,K) = \mathcal{S}(\bar{U},\bar{K})$. Then we obtain that $(u-U, k-K)$ satisfies that

$$\begin{cases} \partial_t^\alpha (u-U)(t,x) + A[u-U](t,x) = \left(\bar{k}-\bar{K}\right) * \bar{u} + \left(\bar{u}-\bar{U}\right) * \bar{K}, \quad (t,x) \in Q_0^T, \\ (u-U)(T,x) - \beta(u-U)(0,x) = 0, \quad x \in \Omega \\ (u-U)(t,x) = 0, \quad (t,x) \in \Sigma_0^T \end{cases} \tag{3.10}$$

and

$$k - K = \frac{1}{h(0)}\left(D_t\mathcal{N}[u,\bar{l}] - D_t\mathcal{N}[U,\bar{L}]\right), \qquad 0 \le t \le T, \tag{3.11}$$

where $\bar{L}(t) = \int_0^t \bar{K}(\tau)d\tau$.

Again by Lemma 2.3, (3.3) and the Young inequality for the convolution, we get

$$\begin{aligned}
\|u-U\|_{X_0^T} \le CT^{\alpha\varepsilon}\Big(&\|(\bar{k}-\bar{K}) * \bar{u}\|_{C([0,T],\mathcal{D}(A^\gamma))} + \|(\bar{u}-\bar{U}) * \bar{K}\|_{C([0,T],\mathcal{D}(A^\gamma))}\Big) \\
&+T^{\alpha\varepsilon}\|(\bar{k}(t)-\bar{K}(t))\bar{u}(0,x)\|_{L^1(0,T;L^2(\Omega))} + T^{\alpha\varepsilon}\|(\bar{k}-\bar{K}) * \bar{u}_t\|_{L^1(0,T;L^2(\Omega))} \\
&+T^{\alpha\varepsilon}\|(\bar{u}(0,x)-\bar{U}(0,x))\bar{K}(t)\|_{L^1(0,T;L^2(\Omega))} + T^{\alpha\varepsilon}\|(\bar{u}-\bar{U})_t * \bar{K}\|_{L^1(0,T;L^2(\Omega))} \\
\le CT^{\alpha\varepsilon}\Big(&\rho\|\bar{k}-\bar{K}\|_{L^1(0,T)} + \rho\|\bar{u}-\bar{U}\|_{C([0,T];D(A^\gamma))}\Big) + T^{\alpha\varepsilon}\|\bar{k}-\bar{K}\|_{L^1(0,T)}\|\bar{u}(0,\cdot)\|_{L^2(\Omega)} \\
+T^{\alpha\varepsilon}\rho\|\bar{k}-\bar{K}\|_{L^1(0,T)} &+ \lambda_1^{-\gamma}T^{\alpha\varepsilon}\rho\|\bar{u}(0,x)-\bar{U}(0,x)\|_{D(A^\gamma)} + T^{\alpha\varepsilon}\rho\|(\bar{u}-\bar{U})_t\|_{L^1(0,T;L^2(\Omega))} \\
\le CT^{\alpha\varepsilon}\rho\Big(&\|\bar{k}-\bar{K}\|_{L^1(0,T)} + \|\bar{u}-\bar{U}\|_{C([0,T];D(A^\gamma))}\Big) + (1+\lambda_1^{-\gamma})T^{\alpha\varepsilon}\rho\|\bar{k}-\bar{K}\|_{L^1(0,T)} \\
&+\lambda_1^{-\gamma}T^{\alpha\varepsilon}\rho\|\bar{u}-\bar{U}\|_{C([0,T];D(A^\gamma))} + T^{\alpha\varepsilon}\rho\|(\bar{u}-\bar{U})_t\|_{L^1(0,T;L^2(\Omega))} \\
&\le C\rho T^{\alpha\varepsilon}\Big(\|\bar{k}-\bar{K}\|_{L^1(0,T)} + \|\bar{u}-\bar{U}\|_{X_0^T}\Big).
\end{aligned} \tag{3.12}$$

Moreover, from (2.27) and (C5), we can easily see that

$$\begin{aligned}
D_t\mathcal{N}[u,\bar{l}] - D_t\mathcal{N}[U,\bar{L}] &= \int_\Omega \phi(x)(A[u_t]-A[U_t])(t,x)dx - (\bar{k}-\bar{K}) * h' \\
&= \int_\Omega A[\phi](u-U)_t(t,x)dx - (\bar{k}-\bar{K}) * h'.
\end{aligned} \tag{3.13}$$

The same as (2.33), it follows that

$$\|D_t\mathcal{N}[u,\bar{l}] - D_t\mathcal{N}[U,\bar{L}]\|_{L^1(0,T)} \le \|\phi\|_{H^2(\Omega)}\|(u-U)_t\|_{L^1(0,T;L^2(\Omega))}$$

$$+T\|\bar{k}-\bar{K}\|_{L^1(0,T)}\|h'\|_{C[0,T]} \leq C\left(\|(u-U)_t\|_{L^1(0,T;L^2(\Omega))}+T\|\bar{k}-\bar{K}\|_{L^1(0,T)}\right). \quad (3.14)$$

Then, from (3.11), together with (3.14), we drive

$$\|k-K\|_{L^1(0,T)} \leq C\left(\|u-U\|_{X_0^T}+T\|\bar{k}-\bar{K}\|_{L^1(0,T)}\right). \quad (3.15)$$

From (3.12) and (3.15)

$$\|(u-U,k-K)\|_{Y_0^T} \leq C\Big(T+\rho T^{\alpha\varepsilon}\Big)\|\bar{k}-\bar{K}\|_{L^1(0,T)}+C\rho T^{\alpha\varepsilon}\|\bar{u}-\bar{U}\|_{X_0^T}$$
$$\leq C(T+\rho T^{\alpha\varepsilon})\|(\bar{u}-\bar{U},\bar{k}-\bar{K})\|_{Y_0^T}. \quad (3.16)$$

Therefore we can choose sufficiently small τ_2 such that

$$CT+\rho T^{\alpha\varepsilon} \leq \frac{1}{2}$$

then, for all $T \in (0,\tau_2]$ we have

$$\|(u-U,k-K)\|_{Y_0^T} \leq \frac{1}{2}\|(\bar{u}-\bar{U},\bar{k}-\bar{K})\|_{Y_0^T}. \quad (3.17)$$

Estimates (3.9) and (3.17) show that $\mathcal{S}$ is a contraction map on $B_{\rho,T}$ for all $T \in (0,\tau]$, if we choose $\tau \leq \min\{1,\tau_1,\tau_2\}$. So, the proof is complete.

To prove the main result, we should prove the following assertion.

Lemma 3.2. *Under conditions (C1)-(C5), for given measurement data $h(t)$ in (1.4), if the inverse problem (1.1)-(1.4) has two solutions $(u_i,k_i) \in Y_0^T$ $(i=1,2)$ for any time $T>0$, then $(u_1,k_1)=(u_2,k_2)$ in $\bar{Q}_0^T$, where $\bar{Q}_0^T=\bar{\Omega}\times[0,T]$.*

According to the Remark 2.2, we know that (2.22)-(2.23) is equivalent to (1.1)-(1.4). So, in Lemma 4.2 we discuss the global uniqueness of the inverse problem (2.22)-(2.23).

Proof

Given any time T, let $(u_i,k_i)\,(i=1,2)$ be two solutions to the inverse problem (2.22)-(2.23) in $[0,T]$ with the regularity $(u_i,k_i) \in Y_0^T$. This implies

$$\|(u_i,k_i)\|_{Y_0^T} \leq C^*, \quad i=1,2, \quad (3.18)$$

where C^* is depending on α, T, initial data φ and ψ, the known function f and measurement data h.

Let

$$\tilde{u}=u_1-u_2, \quad \tilde{k}=k_1-k_2.$$

Then $(\tilde{u},\tilde{k})$ satisfies

$$\begin{cases} \partial_t^\alpha \tilde{u}+A\tilde{u}=k_1*\tilde{u}+\tilde{k}*u_2, & (t,x)\in Q_0^T, \\ \tilde{u}(T,x)-\beta\tilde{u}(0,x)=0, & x\in\Omega, \\ \tilde{u}(t,x)=0, & (t,x)\in\Sigma_0^T, \end{cases} \quad (3.19)$$

and

$$\tilde{k}(t)=\frac{1}{h(0)}\mathrm{D}_t\left(\mathcal{L}[A\tilde{u}]-\tilde{l}*h'\right), \quad (3.20)$$

where $\tilde{l}(t) = (l_1 - l_2)(t)$ and the functions $l_i\,(i = 1, 2)$ satisfy $l_i(t) = \int_0^t k_i(s)ds$. We have to show

$$\|(\tilde{u}, \tilde{k})\|_{Y_0^T} = 0. \tag{3.21}$$

Define

$$\sigma = \inf\Big\{t \in (0, T] : \|(\tilde{u}, \tilde{k})\|_{Y_0^t} > 0\Big\}. \tag{3.22}$$

It suffices to prove that $\sigma = T$. If (3.22) is not true, then it is obvious that σ is well-defined and satisfies $\sigma < T$. Choose ϵ such that $0 < \epsilon < T - \sigma$.

Further, by (2.12), for $(t, x) \in Q_\sigma^{\sigma+\epsilon}$ we can write the solution $\tilde{u}$ as

$$\tilde{u}(x, t) = \int_0^t (t-s)^{\alpha-1}\left(\sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n(t-s)^\alpha)(\tilde{F}(s,\cdot), e_n)e_n(x)\right)ds$$

$$-\int_0^T (T-s)^{\alpha-1}\left(\sum_{n=1}^{\infty}\frac{E_{\alpha,1}(-\lambda_n t^\alpha)}{E_{\alpha,1}(-\lambda_n T^\alpha)-\beta}E_{\alpha,\alpha}(-\lambda_n(T-s)^\alpha)(\tilde{F}(s,\cdot), e_n)e_n(x)\right)ds, \tag{3.23}$$

where

$$\tilde{F}(t, x) = k_1 * \tilde{u} + \tilde{k} * u_2.$$

Then similar to the proofs of Lemma 2.3, we have

$$\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} \le C\epsilon^{\alpha\varepsilon}\left(\|\tilde{F}\|_{C([\sigma,\sigma+\epsilon];D(A^\gamma))} + \|\partial_t\tilde{F}\|_{L^1(\sigma,\sigma+\epsilon;L^2(\Omega))}\right). \tag{3.24}$$

and

$$\|\mathrm{D}_t\mathcal{N}[\tilde{u}, \tilde{l}]\|_{L^1(\sigma,\sigma+\epsilon)} \le C\Big(\|\tilde{u}_t\|_{L^1(\sigma,\sigma+\epsilon;L^2(\Omega))} + \epsilon\|\tilde{k}\|_{L^1(\sigma,\sigma+\epsilon)}\Big). \tag{3.25}$$

From the definition of σ, we see that

$$\tilde{u} = \tilde{k} = 0 \quad \text{in} \quad [0, \sigma]. \tag{3.26}$$

By the definition of $\tilde{F}$, and using (2.5), and (2.27), we have

$$\begin{aligned}\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} &\le C\epsilon^{\alpha\varepsilon}\Big(\|k_1 * \tilde{u}\|_{C([\sigma,\sigma+\epsilon];D(A^\gamma))} + \|\tilde{k} * u_2\|_{C([\sigma,\sigma+\epsilon];D(A^\gamma))}\\ &+\|k_1(t)\tilde{u}(0,x)\|_{L^1(\sigma,\sigma+\epsilon;L^2(\Omega))} + \|k_1 * \tilde{u}_t\|_{L^1(\sigma,\sigma+\epsilon;L^2(\Omega))} + \|\tilde{k}(t)u_2(0,x)\|_{L^1(\sigma,\sigma+\epsilon;L^2(\Omega))}\\ &+\|\tilde{k} * u_{2t}\|_{L^1(\sigma,\sigma+\epsilon;L^2(\Omega))}\Big) \le C\epsilon^{\alpha\varepsilon}\Big(C^*\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];D(A^\gamma))} + C^*\|\tilde{k}\|_{L^1(\sigma,\sigma+\epsilon)}\\ &+\lambda_1^{-\gamma}C^*\|\tilde{u}\|_{C([\sigma,\sigma+\epsilon];D(A^\gamma))} + C^*\|\tilde{u}_t\|_{L^1(\sigma,\sigma+\epsilon;L^2(\Omega))}\\ &+C^*\|\tilde{k}\|_{L^1(\sigma,\sigma+\epsilon)} + C^*\lambda_1^{-\frac{1}{\alpha}}\|\tilde{k}\|_{L^1(\sigma,\sigma+\epsilon)}\Big) \le CC^*\epsilon^{\alpha\varepsilon}\|(\tilde{u}, \tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}}.\end{aligned} \tag{3.27}$$

On the other hand, by (3.20) and (3.25), we have

$$\|\tilde{k}\|_{L^1(\sigma,\sigma+\epsilon)} \le C(\|\tilde{u}\|_{X_\sigma^{\sigma+\epsilon}} + \epsilon\|\tilde{k}\|_{L^1(\sigma,\sigma+\epsilon)}). \tag{3.28}$$

Hence, by (3.27) and (3.28) we obtain

$$\|(\tilde{u}, \tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} \le CC^*(\epsilon + \epsilon^{\alpha\varepsilon})\|(\tilde{u}, \tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} \tag{3.29}$$

implying

$$\|(\tilde{u},\tilde{k})\|_{Y_\sigma^{\sigma+\epsilon}} = 0 \tag{3.30}$$

for some sufficiently small positive constant ϵ_0. This means that $(u_1 - u_2, k_1 - k_2)$ vanishes in $[0, \sigma+\epsilon]$, which contradicts with the definition of σ. Therefore (3.21) is proved. From here, we can conclude that

$$(u_1, k_1) = (u_2, k_2) \quad \text{in} \quad [0, T]$$

for any time T. The proof for Lemma 3.2 is complete.

4 Proof of the main result

Now we prove the global solubility Theorem for our inverse problem. More precisely, for every given time $T > 0$, we will prove the existence of solutions to the problem constituted by (2.22), (2.23), an equivalent form of our inverse problem.

To prove main result, we first show that the local solution can be extended to a larger time interval.

Lemma 3.1 ensures that there exists a unique solution $(\hat{u},\hat{k}) \in Y_0^\tau$ to (2.22)-(2.23) for sufficiently small $\tau > 0$. Now we show that the unique solution $(\hat{u},\hat{k})$ in $[0,\tau]$ can be extended to a larger time interval $[\tau, 2\tau]$.

Proof

Rewrite the system of (2.22)-(2.23) as follows:

$$\begin{cases} \partial_t^\alpha u(t,x) + Au(t,x) = \int_0^\tau \hat{k}(s)\hat{u}(t-s,x)ds \\ \qquad\qquad + \int_\tau^t k(s)u(t-s,x)ds + f(t,x), \quad (t,x) \in Q_\tau^T, \\ u(T,x) - \beta u(\tau,x) = \hat{u}(T,x) - \beta\hat{u}(\tau,x), \quad x \in \Omega, \\ u(t,x) = 0, \quad (t,x) \in \Sigma_\tau^T, \end{cases} \tag{4.1}$$

and

$$k(t) = \frac{1}{h(0)} \mathrm{D}_t \mathcal{N}[u,l](t), \quad t \in [\tau, T], \tag{4.2}$$

where

$$\mathcal{N}[u,l](t) = \partial_t^\alpha h(t) + \mathcal{L}[Au](t) - \int_0^\tau \hat{l}(t-s)h'(s)ds - \int_\tau^t l(t-s)h'(s)ds - \mathcal{L}[f](t), \tag{4.3}$$

and $\hat{l}(t) = \int_0^t \hat{k}(s)ds$.

It is enough to show that (4.1)-(4.2) has a solution $(u(t,x), k(t)) \in Y_\tau^T$. Then (u,k) defined by

$$(u,k) = \begin{cases} (\hat{u},\hat{k}), & t \in [0,\tau], \\ (u,k), & t \in [\tau, 2\tau], \end{cases} \tag{4.4}$$

is an extension of $(\hat{u},\hat{k})$ in $[\tau, T]$, which has the regularity $(u,k) \in Y_\tau^T$. Furthermore, from the global uniqueness result in Lemma 3.2, it follows that (u,k) given by (4.4) is the unique solution to the problem constituted by (2.22)-(2.23) in $t \in [0,\tau]$. This shows that the extension is unique.

We repeat a similar fixed-pointed argument to prove the existence of (u,k). Define an operator

$$\tilde{\mathcal{S}} : \tilde{B}_{\tilde{\rho},T} \to Y_\tau^T, \qquad (\bar{u},\bar{k}) \to (u,k) \tag{4.5}$$

with $(\bar{u},\bar{k}) \in \tilde{B}_{\tilde{\rho},T}$, where

$$\tilde{B}_{\tilde{\rho},T} = \big\{(\bar{u},\bar{k}) \in Y_\tau^T : \ \bar{u}(T,x) - \beta\bar{u}(\tau,x) = \hat{u}(x,T) - \beta\hat{u}(\tau,x), \ x \in \Omega$$

$$\bar{u}(t,x) = 0, \ (t,x) \in \Sigma_\tau^T, \ \|\bar{u}\|_{X_\tau^T} + \|\bar{k}\|_{L^1(\tau,T)} \le \tilde{\rho}\big\}.$$

Here u is the unique solution of the initial and boundary value problem (4.1). Furthermore, k is the solution of (4.2) in terms of u. Additionally, we have $\hat{u}(\tau,\cdot) \in D(A^{\gamma_0})$. In fact, $\hat{u}(\tau,x)$ can be written as follows:

$$\hat{u}(\tau,x) = \sum_{n=1}^{\infty} \frac{\varphi_n}{E_{\alpha,1}(-\lambda_n\tau^\alpha) - \beta} E_{\alpha,1}(-\lambda_n\tau^\alpha)e_n(x)$$

$$-\sum_{n=1}^{\infty} \frac{E_{\alpha,1}(-\lambda_n\tau^\alpha)}{E_{\alpha,1}(-\lambda_nT^\alpha) - \beta}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-s)^\alpha)(F(s,\cdot),e_n)ds \cdot e_n(x)$$

$$+\sum_{n=1}^{\infty} \int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(\tau-s)^\alpha)(F(s,\cdot),e_n)ds \cdot e_n(x), \tag{4.6}$$

with $F(t,x) = \hat{k} * \hat{u} + f \in C([0,\tau]; D(A^\gamma))$ such that $\|F\|_{C([0,\tau];D(A^\gamma))} \le C(\rho,f)$. Then, again using Proposition 1.1, (1.7), and (C5), we get

$$\|\hat{u}(\tau,\cdot)\|^2_{D(A^{\gamma_0})} \le \sum_{n=1}^{\infty} \lambda_n^{2\gamma_0}\left|\frac{\varphi_n}{E_{\alpha,1}(-\lambda_n\tau^\alpha) - \beta}E_{\alpha,1}(-\lambda_n\tau^\alpha)\right|^2$$

$$+\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0}\left|\frac{E_{\alpha,1}(-\lambda_n\tau^\alpha)}{E_{\alpha,1}(-\lambda_nT^\alpha) - \beta}\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-s)^\alpha)(F(s,\cdot),e_n)ds\right|^2$$

$$+\sum_{n=1}^{\infty} \lambda_n^{2\gamma_0}\left|\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(\tau-s)^\alpha)(F(s,\cdot),e_n)ds\right|^2$$

$$\le C_\beta^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0}|\varphi_n|^2 + C_\beta^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0}\left|\int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-s)^\alpha)(F(s,\cdot),e_n)ds\right|^2$$

$$+\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0}\max_{0\le s\le\tau}|(F(s,\cdot),e_n)|^2\left|\int_0^\tau (\tau-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(\tau-s)^\alpha)ds\right|^2$$

$$\le C_\beta^2\|\varphi\|^2_{D(A^{\gamma_0})} + C_\beta^2\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0}\max_{0\le s\le T}|(F(s,\cdot),e_n)|^2\left|\int_0^T \lambda_n^{\varepsilon-1}s^{\alpha\varepsilon-1}ds\right|^2$$

$$+\sum_{n=1}^{\infty}\lambda_n^{2\gamma_0}\max_{0\le s\le\tau}|(F(s,\cdot),e_n)|^2\left|\int_0^\tau \lambda_n^{\varepsilon-1}s^{\alpha\varepsilon-1}ds\right|^2$$

$$\leq C_\beta^2\|\varphi\|_{D(A^{\gamma_0})}^2+\frac{C_\beta^2}{(\alpha\varepsilon)^2}T^{2\alpha\varepsilon}\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\max_{0\leq s\leq T}|(F(s,\cdot),e_n)|^2\lambda_n^{-2(\gamma-\gamma_0-\varepsilon+1)}$$

$$+\frac{C_\beta^2}{(\alpha\varepsilon)^2}\tau^{2\alpha\varepsilon}\sum_{n=1}^{\infty}\lambda_n^{2\gamma}\max_{0\leq s\leq \tau}|(F(s,\cdot),e_n)|^2\lambda_n^{-2(\gamma-\gamma_0-\varepsilon+1)}. \tag{4.7}$$

By $\gamma>\gamma_0+\varepsilon-1$, we have $2\gamma-2\gamma_0+2\varepsilon-2>0$. Thus,

$$\|\hat u(\tau,\cdot)\|_{D(A^{\gamma_0})}\leq C_1\|\varphi\|_{D(A^{\gamma_0})}+$$

$$+C_2\Big(T^{\alpha\varepsilon}\|F\|_{C([0,T];D(A^{\gamma}))}+\tau^{\alpha\varepsilon}\|F\|_{C([0,\tau];D(A^{\gamma}))}\Big). \tag{4.8}$$

hereinafter C_i are constants depending on $\alpha,\beta,\varepsilon,\lambda_1,\gamma_0$ and given functions, but independent of T. Further, similarly as (2.15) for $t\in[\tau,T]$, we have

$$\|u\|_{C([\tau,T];D(A^{\gamma}))}\leq C_3\big(\|\hat u(T,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat u(\tau,\cdot)\|_{D(A^{\gamma_0})}\big)$$

$$+C_4(T-\tau)^{\alpha\varepsilon}\|\tilde F\|_{C([\tau,T];D(A^{\gamma}))}+C_5T^{\alpha\varepsilon}\|\tilde F\|_{C([0,T];D(A^{\gamma}))}, \tag{4.9}$$

where

$$\tilde F(t,x)=\int_0^\tau\hat k(s)\hat u(t-s,x)ds+\int_\tau^t k(s)u(t-s,x)ds+f(t,x). \tag{4.10}$$

As above, we get

$$\|u_t\|_{L^1(\tau,T;L^2(\Omega))}\leq C_6(T-\tau)^{\alpha\varepsilon}\big(\|\hat u(T,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat u(\tau,\cdot)\|_{D(A^{\gamma_0})}\big)$$

$$+C_7(T-\tau)^{\alpha\varepsilon}\|\tilde F\|_{C([0,T];D(A^{\gamma}))}+C_7(T-\tau)^{\alpha\varepsilon}\|\tilde F(\tau,\cdot)\|_{D(A^{\gamma})}$$

$$+C_5(T-\tau)^{\alpha\varepsilon}\|\partial_t\tilde F\|_{L^1(\tau,T;L^2(\Omega))}. \tag{4.11}$$

Note that $\tau\leq t\leq T$. Then by (4.10), we have

$$\left\|\int_0^\tau\hat k(s)\hat u(t-s,x)ds+\int_\tau^t\bar k(s)\bar u(t-s,x)ds+f(t,x)\right\|_{C([\tau,T];D(A^{\gamma}))}$$

$$\leq\|\hat k\|_{L^1(0,\tau)}\|\hat u\|_{C([0,\tau];D(A^{\gamma}))}+\|\bar k\|_{L^1(\tau,T)}\|\bar u\|_{C([\tau,T];D(A^{\gamma}))}+\|f\|_{C([\tau,T];D(A^{\gamma}))}. \tag{4.12}$$

Differentiating (4.10) concerning t, we have

$$\partial_t\tilde F(t,x)=\int_0^\tau\hat k(s)\hat u_t(t-s,x)ds+\bar k(t)\bar u(\tau,x)+\int_\tau^t\bar k(s)\bar u_t(t-s,x)ds+\partial_tf(t,x).$$

As (4.12), we have

$$\|\partial_tF\|_{L^1(\tau,T;L^2(\Omega))}\leq\|\hat k\|_{L^1(0,\tau)}\|\hat u_t\|_{L^1(0,\tau;L^2(\Omega))}+\|\bar k\|_{L^1(\tau,T)}\|\bar u(\tau,\cdot)\|_{L^2(\Omega)}$$

$$+\|\bar k\|_{L^1(\tau,T)}\|\bar u_t\|_{L^1(\tau,T;L^2(\Omega))}+\|\partial_tf\|_{L^1(\tau,T;L^2(\Omega))}. \tag{4.13}$$

So, given (4.9) and (4.11) taking into account (4.12), and (4.13), we have

$$\|u\|_{X_\tau^T}\leq C_8\Big(\|\hat u(\tau,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat u(T,\cdot)\|_{D(A^{\gamma_0})}\Big)$$

$$+C_9(T-\tau)^{\alpha\varepsilon})\Big(1+\|\hat{u}(\tau,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat{u}(T,\cdot)\|_{D(A^{\gamma_0})}\Big)+C_{10}T^{\alpha\varepsilon}. \tag{4.14}$$

On the other hand, by (4.2), we have

$$\|k\|_{L^1(\tau,T)} \leq C_{11}(T-\tau)+\tilde{\rho}T+C_{12}(T-\tau)+C_{13}\|u_t\|_{L^1(\tau,T)}. \tag{4.15}$$

Hence, by (4.14) and (4.15), we have

$$\begin{aligned}\|\tilde{\mathcal{S}}(\bar{u},\bar{k})\|_{Y_\tau^T} \leq C_{14}\Big(\|\hat{u}(\tau,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat{u}(T,\cdot)\|_{D(A^{\gamma_0})}\Big)+C_{11}\\ +C_{15}T^{\alpha\varepsilon}+\tilde{\rho}T+C(h)(T-\tau)\\ +C_{16}(T-\tau)^{\alpha\varepsilon})\Big(1+\|\hat{u}(\tau,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat{u}(T,\cdot)\|_{D(A^{\gamma_0})}\Big),\end{aligned} \tag{4.16}$$

and by similar calculations to (3.17), we have

$$\|\tilde{\mathcal{S}}(\bar{u}_1,\bar{k}_1)-\tilde{\mathcal{S}}(\bar{u}_2,\bar{k}_2)\|_{Y_\tau^T} \leq C_{17}\Big[T^{\alpha\varepsilon}+T-\tau+(T-\tau)^{\alpha\varepsilon}\Big]\|(\bar{u}_1-\bar{u}_2,\bar{k}_1-\bar{k}_2)\|_{Y_\tau^T}. \tag{4.17}$$

We fix $\tilde{\rho}$ such that $\tilde{\rho} \geq \rho$ and

$$C_{14}\Big(\|\hat{u}(\tau,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat{u}(T,\cdot)\|_{D(A^{\gamma_0})}\Big)+C_{11} \leq \frac{\tilde{\rho}}{2}.$$

It is easy to see that if we choose $\tilde{\rho}$ larger, then we could get large $T-\tau$ to satisfy

$$\begin{aligned}C_{16}T^{\alpha\varepsilon}+\tilde{\rho}T+C_{12}(T-\tau)\\ +C_{16}(T-\tau)^{\alpha\varepsilon})\Big(1+\|\hat{u}(\tau,\cdot)\|_{D(A^{\gamma_0})}+|\beta|\|\hat{u}(T,\cdot)\|_{D(A^{\gamma_0})}\Big) \leq \frac{\tilde{\rho}}{2}.\end{aligned} \tag{4.18}$$

Furthermore noticing that (4.18) and (3.16) have the same structure, we can choose $T-\tau=\tau$ to satisfy (4.18), which yields $\|\tilde{S}(\bar{u},\bar{k})\|_{Y_\tau^T} \leq \tilde{\rho}$, i.e. $\tilde{\mathcal{S}}(\tilde{B}_{\tilde{\rho},T}) \subset \tilde{B}_{\tilde{\rho},T}$. Additionally,

$$\|\tilde{\mathcal{S}}(\bar{u}_1,\bar{k}_1)-\tilde{\mathcal{S}}(\bar{u}_2,\bar{k}_2)\|_{Y_\tau^T} \leq \frac{1}{2}\|(\bar{u}_1-\bar{u}_2,\bar{k}_1-\bar{k}_2)\|_{Y_\tau^T} \tag{4.19}$$

for $T=2\tau$, because (4.17) is the same as (3.16), if we replace T in (3.20) by $T-\tau$. Hence we prove that $\tilde{\mathcal{S}}$ is a contraction operator on $\tilde{B}_{\tilde{\rho},T}$ for $T=2\tau$.

Repeating the extension process limited times, we could obtain a solution $(u,k) \in X_0^T \times L^1(0,T)$ of the inverse problem (2.22) and (2.23) for any T. Lemma 3.5 shows that the inverse problem (2.22) and (2.23) is equivalent to our inverse problem. Consequently, the inverse problem (1.1)-(1.4) also admits a unique solution (u,k) in the space $X_0^T \times L^1(0,T)$ for any T.

Conclusions

In this paper, we proved a global uniqueness and the existence of the weak solution to a nonlocal initial and boundary value problem for a time-fractional integro-differential diffusion equation. Firstly, the backward problem was studied and given an auxiliary inverse problem to the original one. In Lemma 2.5, we ensured that the fixed point method to solve our inverse problem can't work. Therefore, the well-posedness of the inverse problem was shown by using the iteration method. Consequently, we showed the global solubility theorem.

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