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FADDEEV EIGENFUNCTIONS FOR MULTIPOINT POTENTIALS [1]

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Abstract We present explicit formulae for the Faddeev eigenfunctions and related generalized scattering data for multipoint potentials in two and three dimensions. For single point potentials in 3D such formulae were obtained in an old unpublished work of L.D. Faddeev. For single point potentials in 2D such formulae were given recently in [11].

Key words: Schrödinger operator, Faddeev eigenfunctions, exact solutions, point scatterers.

AMS Mathematics Subject Classification: 81Q05, 81U15, 35C05, 47A40

1 Introduction

Consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \quad d = 2, 3, \tag{1}$$

where $v(x)$ is a real-valued sufficiently regular function on $\mathbb{R}^d$ with sufficient decay at infinity.

Let us recall that the classical scattering eigenfunctions ψ^+ for (1) are specified by the following asymptotics as $|x| \to \infty$:

$$\psi^+ = e^{ikx} - i\pi\sqrt{2\pi}e^{-\frac{i\pi}{4}} f\left(k, |k|\frac{x}{|x|}\right) \frac{e^{i|k||x|}}{\sqrt{|k||x|}} + o\left(\frac{1}{\sqrt{|x|}}\right), \qquad d = 2, \tag{2}$$

$$\psi^+ = e^{ikx} - 2\pi^2 f\left(k, |k|\frac{x}{|x|}\right) \frac{e^{i|k||x|}}{|x|} + o\left(\frac{1}{|x|}\right), \qquad d = 3, \tag{3}$$

$x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E > 0$, where a priori unknown function $f(k,l)$, $k,l \in \mathbb{R}^d$, $k^2 = l^2 = E$, arising in (2), (3), is the classical scattering amplitude for (1). In addition, we consider the Faddeev eigenfunctions ψ for (1) specified by

$$\psi = e^{ikx}\left(1 + o(1)\right) \quad \text{as} \quad |x| \to \infty, \tag{4}$$

$x \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\operatorname{Im} k \neq 0$, $k^2 = k_1^2 + \ldots + k_d^2 = E$; see [6], [15], [9]. The Faddeev generalized scattering data h arise in more precise version of the expansion (4) (see also

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formulae (10)-(15)). The Faddeev eigenfunctions have very rich analytical properties and are quite important for inverse scattering (see, for example, [7], [13], [9]).

In the present article we consider equation (1), where $v(x)$ is a finite sum of point potentials in two or three dimensions (see [5], [1], [2] and references therein). We will write these potentials as:

$$v(x) = \sum_{j=1}^{n} \varepsilon_j \delta(x - z_j), \tag{5}$$

but the precise sense of these potentials will be specified below (see Section 3) and, strictly speaking, $\delta(x)$ is not the standard Dirac delta-function (in the physical literature the term renormalized δ-function is used).

It is known that for these multipoint potentials the classical scattering eigenfunctions ψ^+ and the related scattering amplitude f can be naturally defined and can be given by explicit formulae (see [1] and references therein). In addition, for single point potentials explicit formulae for the Faddeev eigenfunctions ψ and related generalized scattering amplitude h were obtained in an old unpublished work by L.D. Faddeev for $d = 3$ and in [11] for $d = 2$.

In the present article we give explicit formulae for the Faddeev functions ψ and h for multipoint potentials in the general case for real energies in two and three dimensions (see Theorem 3.1 from the Section 3). Let us point out that our formulae for ψ and h involve the values of the Faddeev Green function G for the Helmholtz equation, where

$$G(x,k) = -\frac{1}{(2\pi)^d}\, e^{ikx} \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi, \tag{6}$$

$$(\Delta + k^2) G(x,k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d, \quad \operatorname{Im} k \neq 0. \tag{7}$$

In the present article we consider $G(x,k)$ as some known special function.

In addition, basic formulae and equations of monochromatic inverse scattering (i.e. inverse scattering at a fixed energy E), derived for sufficiently regular potentials v, remain valid for the Faddeev functions ψ and h of Theorem 3.1. Thus, basic formulae and equations of monochromatic inverse scattering are illustrated by explicit examples related to multipoint potentials. We think that the results of the present work can be used, in particular, for testing different monochromatic inverse scattering algorithms based on properties of the Faddeev functions ψ and h (see [3] as a work in this direction).

It it interesting to note also that explicit formulae for $\psi = \psi(x,k)$ and $h = h(k,l)$ for multipoint potentials show new qualitative effects in comparison with the one-point case. In particular, the Faddeev eigenfunctions $\psi = \psi(x,k)$ for 2-point potentials in 3D may have singularities for real momenta k, in contrast with the one-point potentials in 3D (see Statement 3.1).

Besides, functions ψ and h of Theorem 3.1 for $d = 2$ illustrate a very rich family of 2D potentials with spectral singularities in the complex domain (i.e. with singularities for complex k, l). Let us recall that monochromatic 2D inverse scattering is well-developed only under the assumption that such singularities are absent at fixed energy (see [12]and [11] for additional discussion in this connection). We hope that the aforementioned examples and quite different examples from [8], [18] and recent examples from [14] will help to find correct analytic formulation of monochromatic inverse scattering in two dimensions in the presence of spectral singularities.

2 Some preliminaries

It is convenient to write

$$\psi = e^{ikx}\mu, \tag{8}$$

where ψ solves (1), (4) and μ solves

$$-\Delta\mu - 2ik\nabla\mu + v(x)\mu = 0, \quad k \in \mathbb{C}^d, \ k^2 = E. \tag{9}$$

In addition, to relate eigenfunctions and scattering data it is convenient to use the following presentations, used, for example, in [17] for regular potentials:

$$\mu^+(x,k) = 1 - \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}F(k,-\xi)}{\xi^2 + 2(k+i0k)\xi} d\xi, \quad k \in \mathbb{R}^d\backslash 0, \tag{10}$$

$$\mu_\gamma(x,k) = 1 - \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}H_\gamma(k,-\xi)}{\xi^2 + 2(k+i0\gamma)\xi} d\xi, \quad k \in \mathbb{R}^d\backslash 0, \ \gamma \in S^{d-1}, \tag{11}$$

$$\mu(x,k) = 1 - \int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}H(k,-\xi)}{\xi^2 + 2k\xi} d\xi, \quad k \in \mathbb{C}^d, \ \operatorname{Im} k \neq 0, \tag{12}$$

where $\psi^+ = e^{ikx}\mu^+$ are the eigenfunctions specified by (2), (3), $\psi = e^{ikx}\mu$ are the eigenfunctions specified by (4), $\mu_\gamma(x,k) = \mu(x, k+i0\gamma)$, $k \in \mathbb{R}^d\backslash 0$.

The following formulae hold:

$$f(k,l) = F(k, k-l), \quad k,l \in \mathbb{R}^d, \ k^2 = l^2 = E > 0, \tag{13}$$

$$h_\gamma(k,l) = H_\gamma(k,k-l), \quad k,l \in \mathbb{R}^d, \ k^2 = l^2 = E > 0, \ \gamma \in S^{d-1}, \tag{14}$$

$$h(k,l) = H(k,k-l), \quad k,l \in \mathbb{C}^d, \ \operatorname{Im} k = \operatorname{Im} l \neq 0, \ k^2 = l^2 = E, \tag{15}$$

where f is the classical scattering amplitude of (2), (3), h_γ, h are the Faddeev generalized scattering data of [7].

We recall also that for regular real-valued potentials the following formulae hold (at least outside of the singularities of the Faddeev functions in spectral parameter k):

$$\frac{\partial}{\partial \bar{k}_j}\psi(x,k) = -2\pi \int\limits_{\mathbb{R}^d} \xi_j H(k,-\xi)\psi(x,k+\xi)\delta(\xi^2 + 2k\xi)d\xi, \tag{16}$$

$$\frac{\partial}{\partial \bar{k}_j}H(k,p) = -2\pi \int\limits_{\mathbb{R}^d} \xi_j H(k,-\xi)H(k+\xi,p+\xi)\delta(\xi^2 + 2k\xi)d\xi, \tag{17}$$

$j = 1,\ldots,d$, $k \in \mathbb{C}^d\backslash\mathbb{R}^d$, $x,p \in \mathbb{R}^d$,

$$\psi_\gamma(x,k) = \psi^+(x,k) + 2\pi i \int\limits_{\mathbb{R}^d} h_\gamma(k,\xi)\theta((\xi-k)\gamma)\delta(\xi^2 - k^2)\psi^+(x,\xi)d\xi, \tag{18}$$

$$h_\gamma(k,l) = f(k,l) + 2\pi i \int\limits_{\mathbb{R}^d} h_\gamma(k,\xi)\theta((\xi - k)\gamma)\delta(\xi^2 - k^2) f(\xi,l) d\xi, \tag{19}$$

$\gamma \in S^{d-1}, \quad x,k,l \in \mathbb{R}^d, \quad k^2 = l^2,$
where $\delta(t)$ is the Dirac δ-function, $\theta(t)$ is the Heaviside step function;

$$\mu(x,k) \to 1 \ \text{ for } \ |k| \to \infty, \quad x \in \mathbb{R}^d, \tag{20}$$

$$H(k,p) \to \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} v(x) e^{ipx} dx \ \text{ for } \ |k| \to \infty, \quad p \in \mathbb{R}^d, \tag{21}$$

$$|k| = \sqrt{|\operatorname{Re} k|^2 + |\operatorname{Im} k|^2},$$

see [7], [4], [13] and references therein.

Let us define the following varieties:

$$\Sigma_E = \{k \in \mathbb{C}^d : \ k^2 = E\}, \tag{22}$$

$$\Omega_{E,p} = \{k \in \Sigma_E : \ 2kp = p^2\}, \quad \begin{cases} p = 0 & \text{for} \quad d = 2, \\ p \in \mathbb{R}^3 & \text{for} \quad d = 3, \end{cases} \tag{23}$$

$$\Omega_E = \{k \in \Sigma_E, \quad p \in \mathbb{R}^d : \ 2kp = p^2\}, \tag{24}$$

$$\Theta_E = \{k,l \in \mathbb{C}^d : \quad \operatorname{Im} k = \operatorname{Im} l, \quad k^2 = l^2 = E\}. \tag{25}$$

Note that in the present article we consider the Faddeev functions ψ, H, h and ψ_γ, H_γ, h_γ for multipoint potentials for fixed real energies E only, for simplicity. In this connection we consider

$$\psi \ \text{ on } \ \mathbb{R}^d \times (\Sigma_E \backslash \operatorname{Re}\Sigma_E), \quad H \ \text{ on } \ \Omega_E \backslash \operatorname{Re}\Omega_E, \quad h \ \text{ on } \ \Theta_E \backslash \operatorname{Re}\Theta_E,$$

$$\psi_\gamma(x,k), \quad H_\gamma(k,p), \quad h_\gamma(k,l) \ \text{ for}$$

$$\gamma \in S^{d-1}, \quad x,k,p,l \in \mathbb{R}^d, \quad p^2 = 2kp, \quad k^2 = l^2 = E, \quad k\gamma = 0.$$

In addition, we also consider the forms

$$\bar\partial_k \psi = \sum_{j=1}^{d} \frac{\partial}{\partial \bar k_j} \psi(x,k) d\bar k_j, \quad \bar\partial_k H = \sum_{j=1}^{d} \frac{\partial}{\partial \bar k_j} H(k,p) d\bar k_j,$$

on the varieties Σ_E, $\Omega_{E,p}$, respectively, where the $\partial/\partial \bar k_j$ derivatives of ψ, H are given by (16), (17).

In addition, we recall that formulae (16)-(21) give a basis for monochromatic inverse scattering (i.e. inverse scattering at a fixed energy E) for regular potentials in two and three dimensions, see [4], [9], [10], [12], [13], [15], [16], [17].

3 Main results

By analogy with [5] we understand the multipoint potentials $v(x)$ from (5) as a limit for $N \to +\infty$ of non-local potentials

$$V_N(x,x') = \sum_{j=1}^{n} \varepsilon_j(N) u_{j,N}(x) u_{j,N}(x'), \tag{26}$$

where

$$(V_N\,\mu)(x) = \sum_{j=1}^{n} \varepsilon_j(N) \int\limits_{\mathbb{R}^d} u_{j,N}(x) u_{j,N}(x')\mu(x')dx', \tag{27}$$

$$u_{j,N}(x) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \hat{u}_{j,N}(\xi) e^{i\xi x} d\xi, \quad \hat{u}_{j,N}(\xi) = \begin{cases} e^{-i\xi z_j} & |\xi| \le N, \\ 0 & |\xi| > N, \end{cases} \tag{28}$$

$x, x', z_j \in \mathbb{R}^d$, $z_m \neq z_j$ for $m \neq j$, $\varepsilon_j(N)$ are normalizing constant used below in the formulation of Theorem 3.1 and specified by (40) for $d = 3$ and (41) for $d = 2$. It is clear that

$$u_{j,N}(x) = u_{0,N}(x - z_j), \quad \text{where} \quad \hat{u}_{0,N}(\xi) = \begin{cases} 1 & |\xi| \le N, \\ 0 & |\xi| > N. \end{cases}$$

For $v = V_N$ equation (9) has the following explicit Faddeev solutions:

$$\mu_N(x,k) = 1 + \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \tilde{\mu}_N(\xi,k) e^{i\xi x} d\xi, \tag{29}$$

$$\tilde{\mu}_N(\xi,k) = -\frac{\sum\limits_{j=1}^{n} c_{j,N}(k)\hat{u}_{j,N}(\xi)}{\xi^2 + 2k\xi}, \tag{30}$$

$x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, $k \in \mathbb{C}^d$, $\operatorname{Im} k \neq 0$, where $c_N(k) = (c_{1,N}(k), \ldots, c_{n,N}(k))$ is the solution of the following linear equation:

$$A_N(k) c_N(k) = b_N, \tag{31}$$

where $A_N(k)$ is the $n \times n$ matrix and b_N is the n-component vector with the following elements:

$$A_{m,j,N}(k) = \delta_{m,j} + \varepsilon_m(N) \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} \frac{\hat{u}_{m,N}(-\xi)\hat{u}_{j,N}(\xi)}{\xi^2 + 2k\xi} d\xi, \tag{32}$$

$$b_{m,N} = \varepsilon_m(N). \tag{33}$$

In addition, equation (9) has the following classical scattering solutions:

$$\mu_N^+(x,k) = \mu_N(x, k + i0k), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{R}^d \backslash 0, \tag{34}$$

arising from

$$\tilde{\mu}_N^+(\xi,k) = \tilde{\mu}_N(\xi, k + i0k), \quad \xi \in \mathbb{R}^d, \quad k \in \mathbb{R}^d \backslash 0. \tag{35}$$

Let us consider the following Green functions for the operator $\Delta + 2ik\nabla$:

$$g(x,k) = -\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 + 2k\xi} d\xi, \quad x \in \mathbb{R}^d \ \ k \in \mathbb{C}^d, \quad \operatorname{Im} k \neq 0, \tag{36}$$

$$g_\gamma(x,k) = -\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 + 2(k+i0\gamma)\xi} d\xi, \quad x \in \mathbb{R}^d \ \ k \in \mathbb{R}^d\backslash 0, \quad \gamma \in S^{d-1}, \tag{37}$$

$$g^+(x,k) = -\frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} \frac{e^{i\xi x}}{\xi^2 + 2(k+i0k)\xi} d\xi, \quad x \in \mathbb{R}^d \ \ k \in \mathbb{R}^d\backslash 0. \tag{38}$$

One can see that $G(x,k) = e^{ikx} g(x,k)$, where $G(x,k)$ was defined by (6). Note also that for $d = 3$ the Green function $g^+(x,k)$ can be calculated explicitly:

$$g^+(x,k) = -\frac{1}{4\pi}\frac{e^{-ikx}e^{i|k||x|}}{|x|}. \tag{39}$$

Theorem 3.1. *Let d=2, 3,*

$$\varepsilon_j(N) = \alpha_j \left(1 - \frac{\alpha_j N}{2\pi^2}\right)^{-1}, \quad \alpha_j \in \mathbb{R}, \quad j = 1, \ldots, n, \quad \textit{for} \ \ d = 3, \tag{40}$$

$$\varepsilon_j(N) = \alpha_j \left(1 - \frac{\alpha_j}{2\pi}\ln(N)\right)^{-1}, \quad \alpha_j \in \mathbb{R}, \quad j = 1, \ldots, n, \quad \textit{for} \ \ d = 2, \tag{41}$$

Then:

1. *The limiting eigenfunctions*

$$\psi(x,k) = e^{ikx} \lim_{N\to+\infty} \mu_N(x,k), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{C}^d\backslash\mathbb{R}^d, \quad k^2 = E \in \mathbb{R}, \tag{42}$$

are well-defined (at least outside the spectral singularities in k).

2. *The following formulae hold:*

$$\psi(x,k) = e^{ikx}\left[1 + \sum_{j=1}^{n} c_j(k) g(x - z_j, k)\right], \quad k \in \mathbb{C}^d\backslash\mathbb{R}^d, \quad k^2 = E \in \mathbb{R}, \tag{43}$$

where $c(k) = (c_1(k), \ldots, c_n(k))$ is the solution of the following linear equation:

$$\tilde{A}(k)c(k) = \tilde{b}(k), \tag{44}$$

where $\tilde{A}(k)$ is the $n \times n$ matrix, $\tilde{b}(k)$ is the n-component vector with the following elements for $d = 3$:

$$\tilde{A}_{m,j}(k) = \begin{cases} 1, & m = j \\ -\alpha_m \left(1 - \frac{\alpha_m}{4\pi}|\operatorname{Im} k|\right)^{-1} g(z_m - z_j, k), & m \neq j, \end{cases} \tag{45}$$

$$\tilde{b}_m(k) = \alpha_m \left(1 - \frac{\alpha_m}{4\pi}|\operatorname{Im} k|\right)^{-1}; \tag{46}$$

and with the following elements for $d=2$:

$$\tilde{A}_{m,j}(k) = \begin{cases} 1, & m=j \\ -\alpha_m\left(1-\frac{\alpha_m}{2\pi}(\ln(|\operatorname{Re}k|+|\operatorname{Im}k|)\right)^{-1} g(z_m-z_j,k), & m\neq j, \end{cases} \tag{47}$$

$$\tilde{b}_m(k) = \alpha_m\left(1-\frac{\alpha_m}{2\pi}(\ln(|\operatorname{Re}k|+|\operatorname{Im}k|)\right)^{-1}. \tag{48}$$

In addition, for limiting values of ψ the following formulae hold:

$$\psi_\gamma(x,k) = \psi(x,k+i0\gamma) = e^{ikx}\left[1+\sum_{j=1}^{n} c_{\gamma,j}(k)g_\gamma(x-z_j,k)\right], \tag{49}$$

$$x\in\mathbb{R}^d,\quad k\in\mathbb{R}^d\backslash 0,\quad \gamma\in S^{d-1},\quad k\gamma=0,$$

where $c_\gamma(k)=(c_{\gamma,1}(k),\ldots,c_{\gamma,n}(k))$ is the solution of the following linear equation:

$$\tilde{A}_\gamma(k)c_\gamma(k) = \tilde{b}_\gamma(k), \tag{50}$$

where

$$\tilde{A}_\gamma(k) = \tilde{A}(k+i0\gamma),\quad \tilde{b}_\gamma(k) = \tilde{b}(k+i0\gamma). \tag{51}$$

3. *The Faddeev generalized scattering data for the limiting potential $v=\lim_{N\to+\infty} V_N$, associated with the limiting eigenfunctions ψ, ψ_γ, are given by:*

$$h(k,l) = \frac{1}{(2\pi)^d}\sum_{j=1}^{n} c_j(k)e^{i(k-l)z_j}, \tag{52}$$

$$k,l\in\mathbb{C}^d,\quad \operatorname{Im}k=\operatorname{Im}l\neq 0,\quad k^2=l^2=E\in\mathbb{R},$$

where $c_j(k)$ are the same as in (43), (44);

$$h_\gamma(k,l) = \frac{1}{(2\pi)^d}\sum_{j=1}^{n} c_{\gamma,j}(k)e^{i(k-l)z_j}, \tag{53}$$

$$k,l\in\mathbb{R}^d\backslash 0,\quad k^2=l^2=E,\quad \gamma\in S^{d-1},\quad k\gamma=0,$$

where $c_{\gamma,j}(k)$ are the same as in (49), (50).

Note that if $\|\tilde{b}(k)\| = \left(\sum_{j=1}^n |b(k)|^2\right)^{1/2} = \infty$ for some k in formulae (43)-(51) then for such k we understand (43)-(51) as (71), (73)-(75), (85), (87)-(89).

Remark 3.1. *Let the assumptions of Theorem 3.1 be fulfilled. Then:*

1. *For the classical scattering eigenfunctions ψ^+ the following formulae hold:*

$$\psi^+(x,k) = e^{ikx}\left[1+\sum_{j=1}^{n} c_j^+(k)g^+(x-z_j,k)\right], \tag{54}$$

where $c^+(k) = (c_1^+(k), \ldots, c_n^+(k))$ is the solution of the following linear equation:

$$\tilde A^+(k) c^+(k) = \tilde b^+(k), \tag{55}$$

where $\tilde A^+(k)$ is the $n \times n$ matrix, and $\tilde b^+(k)$ is the n-component vector with the following elements for $d = 3$:

$$\tilde A^+_{m,j}(k) = \begin{cases} 1 & m = j \\ -\alpha_m \left(1 + \frac{i\alpha_m}{4\pi}|k|\right)^{-1} g^+(z_m - z_j, k), & m \neq j, \end{cases} \tag{56}$$

$$\tilde b^+_m(k) = \alpha_m \left(1 + \frac{i\alpha_m}{4\pi}|k|\right)^{-1}; \tag{57}$$

and with the following elements for $d = 2$:

$$\tilde A^+_{m,j}(k) = \begin{cases} 1 & m = j \\ -\alpha_m \left(1 + \frac{\alpha_m}{4\pi}(\pi i - 2\ln|k|)\right)^{-1} g^+(z_m - z_j, k), & m \neq j, \end{cases} \tag{58}$$

$$\tilde b^+_m(k) = \alpha_m \left(1 + \frac{\alpha_m}{4\pi}(\pi i - 2\ln|k|)\right)^{-1}; \tag{59}$$

2. For the classical scattering amplitude f the following formula holds:

$$f(k,l) = \frac{1}{(2\pi)^d} \sum_{j=1}^{n} c_j^+(k) e^{i(k-l)z_j}, \tag{60}$$
$$k, l \in \mathbb{R}^d, \quad k^2 = l^2 = E \in \mathbb{R},$$

where $c_j^+(k)$ are the same as in (54), (55).

In a slightly different form formulae (54) - (60) are contained in Section II.1.5 and Chapter II.4 of [1]. In addition, the classical scattering functions ψ^+ and f for $d = 3$ are expressed in terms of elementary functions via (54)- (60).

Proposition 3.1. *Formulae (16),(17) in terms of $\bar\partial_k \mu$, $\bar\partial_k H$, on Σ_E, $\Omega_{E,p}$, formulae (18), (19) with $k\gamma = 0$ and formula (20) for $|\operatorname{Im} k| \to \infty$ are fulfilled for functions $\psi = e^{ikx}\mu$, ψ_γ, ψ^+, h, h_γ of Theorem 3.1, at least for $x \neq z_j$, $j = 1, \ldots, n$.*

Statement 3.1. *Let $d = 3$, $n = 2$, $E = E_{fix} > 0$. Then for appropriate α_1, $\alpha_2 \in \mathbb{R}\backslash 0$, $z_1, z_2 \in \mathbb{R}^3$ there are real spectral singularities $k = k' + i0\gamma'$ with $\gamma' \in S^2$, $k' \in \mathbb{R}^3$, $(k')^2 = E_{fix}$, $k'\gamma' = 0$, of the Faddeev functions $\psi = \psi(x,k)$, $h = h(k,l)$ of Theorem 3.1.*

Remark 3.2. *In connection with Statement 3.1, note that for the case $d = 3$, $n = 1$, studied in the old unpublished work of Faddeev, there are no real spectral singularities of the Faddeev functions ψ, h. In addition, in [11] it was shown that for the case $d = 2$, $n = 1$, $\alpha \in \mathbb{R}\backslash 0$ the Faddeev functions always have some real spectral singularities (see Statement 3.1 of [11] for details).*

Let us recall that $\dim_{\mathbb{C}} \Sigma_E = 1$, $\dim_{\mathbb{R}} \Sigma_E = 2$ for $d = 2$, where Σ_E is defined by (22). In addition, it is known that for a fixed real energy $E = E_{\rm fix}$ the spectral singularities of $\psi = \psi(x, k)$ and $H = H(k, p)$ on $\Sigma_E \backslash \operatorname{Re} \Sigma_E$ are zeroes of a real-valued determinant function $\Delta = \Delta(k)$ (modified Fredholm determinant for linear integral equation for $\psi(\cdot, k)$) for real-valued potentials. Thus, one can expect that these spectral singularities on $\Sigma_{E_{\rm fix}}$ for generic real potentials are either empty or form a family of curves Γ_j, $j = \pm 1, \pm 2, \ldots \pm J$ for $d = 2$. The problem of studying the geometry of these spectral singularities on $\Sigma_{E_{\rm fix}}$ was formulated already in [12]. In addition, it was expected in [12] that the most natural configuration of curves is a "nest"

$$[\Gamma_{-J} \subset \Gamma_{-J+1} \subset \ldots \subset \Gamma_{-1} \subset S^1 \subset \Gamma_1 \subset \ldots \subset \Gamma_J], \tag{61}$$

see [12] for details.

Figures Figure 1–Figure 4 show these spectral singularities for 2-point potentials for some interesting cases for $d = 2$. These figures show that the geometry of the singular curves Γ_j may be different from the "nest".

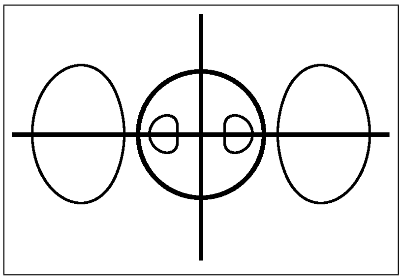

Figure 1
$E = 4, \quad z_2 - z_1 = (0.5\,,\,0),$
$\alpha_1 = 5, \quad \alpha_2 = 6$

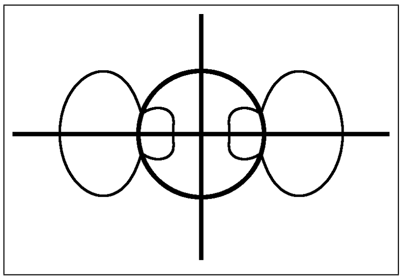

Figure 2
$E = 6, \quad z_2 - z_1 = (0.5\,,\,0),$
$\alpha_1 = 5, \quad \alpha_2 = 6$

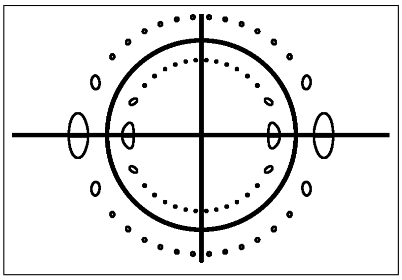

Figure 3
$E = 5, \quad z_2 - z_1 = (10\,,\,0),$
$\alpha_1 = 6, \quad \alpha_2 = 6$

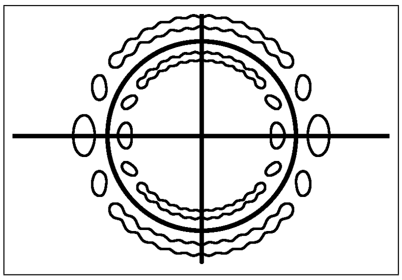

Figure 4
$E = 5, \quad z_2 - z_1 = (10\,,\,0),$
$\alpha_1 = 6, \quad \alpha_2 = 6.8$

In Figures 1-4 the surface $\Sigma_E$for $d = 2$ is shown as $\mathbb{C}\backslash 0$ with the coordinate λ, where the parametrization of Σ_E is given by the formulae:

$$k_1 = \left(\frac{1}{\lambda} + \lambda\right) \frac{\sqrt{E}}{2}, \quad k_2 = \left(\frac{1}{\lambda} - \lambda\right) \frac{i\sqrt{E}}{2}, \qquad \lambda \in \mathbb{C}\backslash 0. \tag{62}$$

The coordinate axes $\operatorname{Im}\lambda = 0$, $\operatorname{Re}\lambda = 0$ and the unit circle $|\lambda| = 1$ in $\mathbb{C}$ are shown in bold. This unit circle corresponds to $\Sigma_E \cap \mathbb{R}^2$, i.e. to real (physical) momenta $k = (k_1, k_2)$. The other black sets inside the rectangles in Figures 1-4 show singular curves Γ_j.

4 Sketch of proofs

To prove Theorem 3.1 we proceed from formulae (28)-(33). We rewrite (31) as

$$\left(I + \Lambda_N^{-1}(k)\, B_N(k)\right) c_N(k) = \Lambda_N^{-1}(k)\, b_N, \tag{63}$$

where $\Lambda_N(k)$ and $B_N(k)$ are the diagonal and off-diagonal parts of $A_N(k)$, respectively. One can see that

$$(\Lambda_N^{-1}(k)\, b_N)_m = \frac{\varepsilon_m(N)}{1 + \varepsilon_m(N)\,\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat u_{m,N}(-\xi)\hat u_{m,N}(\xi)}{\xi^2+2k\xi}d\xi}, \tag{64}$$

$$(\Lambda_N^{-1}(k)\, B_N(k))_{m,j} = (1-\delta_{m,j})\,\frac{\varepsilon_m(N)\,\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat u_{m,N}(-\xi)\hat u_{j,N}(\xi)}{\xi^2+2k\xi}d\xi}{1 + \varepsilon_m(N)\,\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat u_{m,N}(-\xi)\hat u_{m,N}(\xi)}{\xi^2+2k\xi}d\xi}. \tag{65}$$

In addition, for $N \to +\infty$:

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat u_{m,N}(-\xi)\hat u_{j,N}(\xi)}{\xi^2+2k\xi}d\xi \to -g(z_m - z_j, k), \quad j \neq m, \quad \text{for} \quad d = 2, 3, \tag{66}$$

$$\varepsilon_m(N)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat u_{m,N}(-\xi)\hat u_{m,N}(\xi)}{\xi^2+2k\xi}d\xi \to \frac{\alpha_m}{1 - \frac{\alpha_m}{4\pi}|\operatorname{Im} k|} \quad \text{for} \quad d = 3, \tag{67}$$

$$\varepsilon_m(N)\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat u_{m,N}(-\xi)\hat u_{m,N}(\xi)}{\xi^2+2k\xi}d\xi \to \frac{\alpha_m}{1 - \frac{\alpha_m}{2\pi}(\ln(|\operatorname{Re} k| + |\operatorname{Im} k|)} \quad \text{for} \quad d = 2, \tag{68}$$

$k \in \mathbb{C}^d \backslash \mathbb{R}^d$, $k^2 = E \in \mathbb{R}$.

One can see that (66) follows from (36) and the definition of $\hat u_{j,N}$ in (28). In turn, formulae (67), (68) follow from (40), (41), the definition of $\hat u_{j,N}$ and the following asymptotic formulae for $N \to +\infty$:

$$\int\limits_{\xi\in\mathbb{R}^d,\ |\xi|\le N}\frac{1}{\xi^2+2k\xi}d\xi = 4\pi N - 2\pi^2|\operatorname{Im} k| + O(N^{-1}) \quad \text{for} \quad d = 3, \tag{69}$$

$$\int\limits_{\xi\in\mathbb{R}^d,\ |\xi|\le N}\frac{1}{\xi^2+2k\xi}d\xi = 2\pi\ln N - 2\pi\ln(|\operatorname{Re} k| + |\operatorname{Im} k|) + O(N^{-1}) \quad \text{for} \quad d = 2, \tag{70}$$

where $k \in \mathbb{C}^d \backslash \mathbb{R}^d$, $k^2 = E \in \mathbb{R}$.

Formulae (42)-(48) follow from (28)-(30), (63)-(68).

Formulae (49)-(51) follow from (43)-(48).

Formulae (52)-(53) follow from the relations $\psi = e^{ikx}\mu$, $\psi_\gamma = e^{ikx}\mu_\gamma$, and formulae (11), (12), (14), (15), (36),(37), (43), (49).

This completes the sketch of proof of Theorem 3.1.

To prove Proposition 3.1 we rewrite (43)-(48), (52) in the following form:

$$\psi(x,k)=e^{ikx}+\sum_{j=1}^{n}\mathcal{C}_j(k)G(x-z_j,k), \qquad (71)$$

$$H(k,p)=\frac{1}{(2\pi)^d}\sum_{j=1}^{n}\mathcal{C}_j(k)e^{-ikz_j}e^{ipz_j}, \qquad (72)$$

$$\mathcal{A}\mathcal{C}=\mathcal{B}, \qquad (73)$$

$$\begin{array}{llll}\mathcal{A}_{m,m}(k) & = \alpha_m^{-1}-(4\pi)^{-1}|\operatorname{Im}k|, & d=3, & \\ \mathcal{A}_{m,m}(k) & = \alpha_m^{-1}-(2\pi)^{-1}\ln(|\operatorname{Re}k|+|\operatorname{Im}k|), & d=2, & (74)\\ \mathcal{A}_{m,j}(k) & = -G(z_m-z_j,k), & m\neq j, & \end{array}$$

$$\mathcal{B}_m(k)=e^{ikz_m}, \qquad (75)$$

where $k\in\mathbb{C}^d\backslash\mathbb{R}^d$, $k^2=E\in\mathbb{R}$, $p\in\mathbb{R}^d$, $p^2=2kp$, G is defined by (6).

Here

$$\mathcal{C}_j(k)=e^{ikz_j}c_j(k).$$

We recall the formulae (see [13])

$$\frac{\partial}{\partial\bar{k}_j}G(x,k)=-\frac{1}{(2\pi)^{d-1}}\int\limits_{\mathbb{R}^d}\xi_j e^{i(k+\xi)x}\delta(\xi^2+2k\xi)d\xi,\quad j=1,\ldots,d. \qquad (76)$$

$$G(x,k+\xi)=G(x,k),\quad \text{for}\quad \xi\in\mathbb{R}^d,\quad \xi^2+2k\xi=0, \qquad (77)$$

where $k\in\mathbb{C}^d\backslash\mathbb{R}^d$.

We will use also the following formula:

$$\bar{\partial}_k\mathcal{A}_{m,m}(k)=\frac{1}{(2\pi)^{d-1}}\int\limits_{\mathbb{R}^d}\left(\sum_{j=1}^{d}\xi_j d\bar{k}_j\right)\delta(\xi^2+2k\xi)\,d\xi\quad\text{on}\quad\Sigma_E\backslash\operatorname{Re}\Sigma_E,\quad E\in\mathbb{R}. \qquad (78)$$

The proof of the $\bar{\partial}$-equation (16) for $\bar{\partial}_k\psi(x,k)$ on $\Sigma_E\backslash\operatorname{Re}\Sigma_E$ can be sketched as formulae (79)-(84) on $\Sigma_E\backslash\operatorname{Re}\Sigma_E$ as follows.

We have

$$\bar{\partial}_k\psi(x,k)=\sum_{j=1}^{n}\mathcal{C}_j(k)(\bar{\partial}_kG(x-z_j,k))+\sum_{j=1}^{n}(\bar{\partial}_k\mathcal{C}_j(k))G(x-z_j,k). \qquad (79)$$

Using (72), (76) one can see that:

$$\sum_{j=1}^{n}\mathcal{C}_j(k)(\bar{\partial}_kG(x-z_j,k))=-2\pi\int\limits_{\mathbb{R}^d}\left(\sum_{s=1}^{d}\xi_s d\bar{k}_s\right)H(k,-\xi)e^{i(k+\xi)x}\delta(\xi^2+2k\xi)d\xi. \qquad (80)$$

Taking into account (71), (72), (79), (80) one can see that to prove equation (16) it is sufficient to verify the following $\bar{\partial}$-equation:

$$\bar{\partial}_k \mathcal{C}_m(k) = -(2\pi)^{d-1} \int\limits_{\mathbb{R}^d} \left(\sum_{s=1}^{d} \xi_s d\bar{k}_s \right) \left[\sum_{j=1}^{n} \mathcal{C}_j(k) e^{-i(k+\xi)z_j} \mathcal{C}_j(k+\xi) \right] \delta(\xi^2 + 2k\xi) d\xi. \quad (81)$$

In turn, (81) follows form the following formulae:

$$(\bar{\partial}_k \mathcal{C})\, \mathcal{A} + \mathcal{C}\, (\bar{\partial}_k \mathcal{A}) = 0, \quad (82)$$

$$\bar{\partial}_k \mathcal{A}_{m,j}(k) = \frac{1}{(2\pi)^{d-1}} \int\limits_{\mathbb{R}^d} \left(\sum_{s=1}^{d} \xi_s d\bar{k}_s \right) e^{i(k+\xi)z_m} e^{-i(k+\xi)z_j} \delta(\xi^2 + 2k\xi) d\xi, \quad (83)$$

$$(\mathcal{A}^{-1} \bar{\partial}_k \mathcal{A})_{m,j}(k) = \frac{1}{(2\pi)^{d-1}} \int\limits_{\mathbb{R}^d} \left(\sum_{s=1}^{d} \xi_s d\bar{k}_s \right) \mathcal{C}_m(k+\xi) e^{-i(k+\xi)z_j} \delta(\xi^2 + 2k\xi) d\xi. \quad (84)$$

The $\bar{\partial}$-equation (17) for $\bar{\partial}_k H$ on $\Sigma_E \backslash \operatorname{Re} \Sigma_E$ follows from formula (12) and the $\bar{\partial}$-equation (16) for $\bar{\partial}_k \psi$ on $\Sigma_E \backslash \operatorname{Re} \Sigma_E$.

To verify (18) with $k\gamma = 0$ we rewrite (49)-(51), (53) and (54)-(60) in a similar way with (71)-(75):

$$\psi_\gamma(x,k) = e^{ikx} + \sum_{j=1}^{n} \mathcal{C}_{\gamma,j}(k) G_\gamma(x - z_j, k), \quad (85)$$

$$h_\gamma(k,l) = \frac{1}{(2\pi)^d} \sum_{j=1}^{n} \mathcal{C}_{\gamma,j}(k) e^{-ilz_j}, \quad (86)$$

$$\mathcal{A}_\gamma\, \mathcal{C}_\gamma = \mathcal{B}_\gamma, \quad (87)$$

$$\begin{array}{lcll} \mathcal{A}_{\gamma,m,m}(k) & = & \alpha_m^{-1}, & d=3, \\ \mathcal{A}_{\gamma,m,m}(k) & = & \alpha_m^{-1} - (2\pi)^{-1} \ln(|k|), & d=2, \\ \mathcal{A}_{\gamma,m,j}(k) & = & -\,G_\gamma(z_m - z_j, k), & m \neq j, \end{array} \quad (88)$$

$$\mathcal{B}_{\gamma,m}(k) = e^{ikz_m}, \quad (89)$$

where $\gamma \in S^{d-1}$, $k, l \in \mathbb{R}^d \backslash 0$, $k\gamma = 0$, $G_\gamma(x,k) = G(x, k + i0\gamma)$;

$$\psi^+(x,k) = e^{ikx} + \sum_{j=1}^{n} \mathcal{C}_j^+(k) G^+(x - z_j, k), \quad (90)$$

$$f(k,l) = \frac{1}{(2\pi)^d} \sum_{j=1}^{n} \mathcal{C}_j^+(k) e^{-ilz_j}, \quad (91)$$

$$\mathcal{A}^+\, \mathcal{C}^+ = \mathcal{B}^+, \quad (92)$$

$$\begin{aligned}
\mathcal{A}^+_{m,m}(k) &= \alpha_m^{-1} + i(4\pi)^{-1}|k|, && d=3,\\
\mathcal{A}^+_{m,m}(k) &= \alpha_m^{-1} + (4\pi)^{-1}(\pi i - 2\ln(|k|)), && d=2, \qquad (93)\\
\mathcal{A}^+_{m,j}(k) &= -G^+(z_m - z_j, k), && m\neq j,
\end{aligned}$$

$$\mathcal{B}^+_m(k) = e^{ikz_m}, \qquad (94)$$

where $k, l \in \mathbb{R}^d\backslash 0$.

We recall the formula (see [7], [13]):

$$G_\gamma(x,k) = G^+(x,k) + \frac{2\pi i}{(2\pi)^d}\int\limits_{\xi\in\mathbb{R}^d} e^{i\xi x}\delta(\xi^2 - k^2)\theta((\xi-k)\gamma)d\xi, \qquad (95)$$

where $\gamma\in S^{d-1}$, $k\in\mathbb{R}^d\backslash 0$.

We will use also the following formula:

$$\mathcal{A}_{\gamma,m,m}(k) = \mathcal{A}^+_{m,m}(k) - \frac{2\pi i}{(2\pi)^d}\int\limits_{\xi\in\mathbb{R}^d}\delta(\xi^2-k^2)\theta(\xi\gamma)d\xi, \qquad (96)$$

where $\gamma\in S^{d-1}$, $k\in\mathbb{R}^d\backslash 0$, $k\gamma=0$.

One can see that for ψ_γ, ψ^+ of (85), (90) relation (18) with $k\gamma=0$ is reduced to the following two relations:

$$\sum_{j=1}^n \mathcal{C}_{\gamma,j}(k)\left(G_\gamma(x-z_j,k) - G^+(x-z_j,k)\right) = \qquad (97)$$

$$= 2\pi i\int_{\mathbb{R}^d} h_\gamma(k,\xi)e^{i\xi x}\delta(\xi^2-k^2)\theta(\xi\gamma)d\xi,$$

$$\mathcal{C}_{\gamma,j}(k) = \mathcal{C}^+_j(k) + 2\pi i\int_{\mathbb{R}^d} h_\gamma(k,\xi)\delta(\xi^2-k^2)\theta(\xi\gamma)\mathcal{C}^+_j(\xi)d\xi, \qquad (98)$$

where $\gamma\in S^{d-1}$, $k\in\mathbb{R}^d\backslash 0$, $k\gamma=0$.

Relation (97) follows from (95) and (86). Relation (98) follows from the following relations

$$(I + (\mathcal{A}^+)^{-1}(\mathcal{A}_\gamma - \mathcal{A}^+))\mathcal{C}_\gamma = \mathcal{C}^+, \qquad (99)$$

$$(\mathcal{A}_\gamma(k) - \mathcal{A}^+(k))_{m,j} = -\frac{2\pi i}{(2\pi)^d}\int\limits_{\xi\in\mathbb{R}^d} e^{i\xi(z_m - z_j)}\delta(\xi^2-k^2)\theta(\xi\gamma)d\xi, \qquad (100)$$

$$[(\mathcal{A}^+(k))^{-1}(\mathcal{A}_\gamma(k) - \mathcal{A}^+(k))]_{m,j} = -\frac{2\pi i}{(2\pi)^d}\int\limits_{\xi\in\mathbb{R}^d}\mathcal{C}^+_m(\xi)e^{-i\xi z_j}\delta(\xi^2-k^2)\theta(\xi\gamma)d\xi, \qquad (101)$$

and formula (86) for h_γ.

This completes the sketch of proof of the relation (18).

Relation (19) can be obtained using (10), (11), (13), (14), (18).

Formula (20) for $|\operatorname{Im} k|\to\infty$ can be obtained using (43)-(48).

Sketch of proof of Proposition 3.1 is completed.

To prove Statement 3.1 we point out that spectral singularities of ψ, h on Σ_E, $E\in\mathbb{R}$, coincide with the zeroes of $\det\mathcal{A}(k)$, where $\mathcal{A}(k)$ is defined by (74) (we can always assume that all $\alpha_m\neq 0$). For $d=3$, $n=2$ we have that

$$\det\mathcal{A}(k)=\left[\frac{1}{\alpha_1}-\frac{|\operatorname{Im}k|}{4\pi}\right]\cdot\left[\frac{1}{\alpha_2}-\frac{|\operatorname{Im}k|}{4\pi}\right]-G(z_1-z_2,k)\cdot G(z_2-z_1,k). \qquad (102)$$

We recall that $G(x,k)$ is real-valued (see [13]) or, more precisely,

$$G(x,k)=\overline{G(x,k)},\quad k\in\Sigma_E\backslash\operatorname{Re}\Sigma_E,\quad E\in\mathbb{R}. \qquad (103)$$

For $k=k'+i0\gamma'$ of Statement 3.1 formulae (102), (103) take the form:

$$\det\mathcal{A}(k'+i0\gamma')=\frac{1}{\alpha_1\alpha_2}-G_{\gamma'}(z_1-z_2,k')\cdot G_{\gamma'}(z_2-z_1,k'). \qquad (104)$$

$$G_{\gamma'}(x,k')=\overline{G_{\gamma'}(x,k')}. \qquad (105)$$

Therefore, for z_1, z_2 such that $G_{\gamma'}(z_1-z_2,k')\cdot G_{\gamma'}(z_2-z_1,k')\neq 0$ one can always choose α_1, $\alpha_2\in\mathbb{R}$ such that $\det\mathcal{A}(k'+i0\gamma')=0$.

Statement 3.1 is proved.

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