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OPTIMAL CONTROL PROBLEM WITH CONTROLS IN COEFFICIENTS OF QUASILINEAR ELLIPTIC EQUATION

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Abstract The optimal control problem with controlling functions in coefficients of multidimensional quasilinear elliptic type equation is considered. Studding optimal control problem consist the nonlinear phase and integral constraints and nonlinear quality criterion in general integral form. Correctness of this problem is investigated, existence theorem are proved and the necessary condition in the form of the generalized rule of Lagrange multipliers for optimality of solution is established. Found the formula for variation of quality criterion, which may be used in numerical treatments.

Key words: optimal control, controls in coefficients of PDE, quasilinear elliptic equation, correctness, variation methods

AMS Mathematics Subject Classification: 35B37, 49J20, 49J40, 93B30, 93C10, 93C20

1 Introduction

The investigation of the problems of optimal control of systems described by elliptic type PDEs when controlling functions are in coefficients of the equations meets with serious difficulties. These problems usually are strong nonlinear and incorrect [see: 1-3]. Optimal control problem for elliptic type equation with controls in coefficients aroused in optimization of structure of continuous media, designing of constructions, elasticity theories, convection-diffusion processes [4, 5]. At present the problems of optimal control in coefficients of linear elliptic equations has been already considered in [6-13] and other works. Such problems for quasilinear elliptic equations are not have been enough studied [14, 15].

In this paper the optimal control problem with nonlinear criteria of quality for class of quasilinear elliptic equations with controlling functions in coefficients and with nonlinear phase and functional constraints is considered. Correctness of the problem is investigated and the necessary condition of optimality in the form of the generalized rule of Lagrange multipliers is established.

2 Statement of the problem

Let the domain $\Omega \subset E_n (n \geq 2)$ is a full-sphere, a spherical stratum, a parallelepiped or can be transformed to one of these domains by means of regular transformation from $C^2(\bar{\Omega})$, Γ is a continuous Lipschitz boundary of domain Ω, $x = (x_1, \ldots, x_n) \in \bar{\Omega}$ is an arbitrary point. Designations of functional spaces and their norms used in the paper correspond to [16, pp. 27-30]. Below, the positive constants which independent on estimated quantities and admissible controls, are denoted as $M_j, \quad (j = 1, 2, ...)$.

Let the controlling system is described by the following quasilinear elliptic equation

$$-\sum_{i,j=1}^{n} \left(a_{ij}(x, k(x))u_{x_j}\right)_{x_i} + q(x)a(x, u) = f(x), \quad x \in \Omega, \tag{2.1}$$

with boundary condition

$$u(x) = 0, \quad x \in \Gamma, \tag{2.2}$$

where $a_{ij}(x, k) \ \ (i, j = \overline{1, n})$, $a(x, u)$, $f(x)$ are given functions, $k(x) = (k_1(x), \ldots, k_r(x))$, $q(x)$ are the controlling functions. Let $v(x) = (k(x), q(x))$ is a control, $u = u(x, v)$ is a solution of the problem (2.1), (2.2) corresponding to control $v(x)$. Let us introduce the set of admissible controls $V_{ad} = K \times Q$, where

$$K = \{k(x) = (k_1(x), ..., k_r(x)) \in (W^1_\infty(\Omega))^r : 0 < \nu_i \le k_i(x) \le \mu_i \ ,$$

$$\left|k_{ix_j}(x)\right| \le d_i^{(j)} \quad (i = \overline{1, r}; \ \ j = \overline{1, n}), \quad \text{a. e. on } \Omega\}, \tag{2.3}$$

$$Q = \{q(x) \in L_\infty(\Omega) : 0 \le q_0 \le q(x) \le q_1 \quad \text{a. e. on} \quad \Omega\},$$

where $\mu_i \ge \nu_i > 0, \quad d_i^{(j)} > 0 \quad (i = \overline{1, r}, \quad j = \overline{1, n}), \quad q_1 \ge q_0 \ge 0-$ are given numbers, a.e. denotes a property "almost everywhere".

Let us formulate the following optimal control problem: among of all admissible controls $v(x) = (k(x), q(x)) \in V_{ad}$, satisfying constraints

$$J_l(v) = \int_\Omega [F_l(x, u(x, v), u_x(x, v), k(x)) + q(x)G_l(x, u(x, v), u_x(x, v))]dx \le 0, \quad (l = \overline{1, l_0}), \tag{2.4}$$

find a control $v_*(x) = (k_*(x), q_*(x)) \in V_{ad}$, minimizing the functional

$$J_0(v) = \int_\Omega [F_0(x, u(x, v), u_x(x, v), k(x)) + q(x)G_0(x, u(x, v), u_x(x, v))]dx \tag{2.5}$$

where $F_l(x, u, p, k), \quad G_l(x, u, p)) \qquad (l = \overline{0, l_0})$ are given functions its arguments, $p = (p_1, ..., p_n)$, $k = (k_1, ..., k_r)$.

Let us suppose, that following conditions are below satisfied

1) $f(x) \in L_2(\Omega)$; the functions $a_{ij}(x, k)$ and their partial derivatives with respect to $x_m \quad m = \overline{1, n}$ are measurable with respect to $x \in \Omega$ and continuous with respect to $k \in K_0$, where

$$K_0 = \{k = (k_1, ..., k_r) \in E_r : 0 < \nu_i \le k_i \le \mu_i \ \ (i = \overline{1, r})\};$$

2) for almost all $x \in \Omega$ and for all $\xi = (\xi_1, ..., \xi_n) \in E_n$, $k(x) \in K$, the ellipticity inequality

$$\nu \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x, k(x))\xi_i\xi_j \le \mu \sum_{i=1}^{n} \xi_i^2,$$

is valid, $\mu, \nu = const > 0$ and for all $k(x) \in K$ the following inequalities are take place

$$\|a_{ijx_m}(x, k(x))\|_{n+1,\Omega} \le \mu \quad (i, j, m = \overline{1, n});$$

3) the function $a(x, u)$ are measurable with respect to $x \in \Omega$ and continuous with respect to $u \in R$, for almost all $x \in \Omega$ and for all $u_1, u_2 \in R$ the relations

$$a(x, 0) = 0, \quad 0 \le [a(x, u_1) - a(x, u_2](u_1 - u_2) \le L(u_1 - u_2)^2, \ \ L = const > 0$$

are hold.

4) functions $F_l(x, u, p, k)$, $G_l(x, u, p)$ are measurable with respect to $x \in \Omega$ and continuous with respect to $u \in R$, $p \in E_n$, $k \in K_0$; for $n = 2$ and $n = 3$ for any $h > 0$ there exist such functions $\alpha^{(h)}(x)$, $\beta^{(h)}(x) \in L_1(\Omega)$, and constants M_3, $M_4 > 0$, that for almost all $x \in \Omega$, and for all $u \in [-h, h]$, $p \in E_n$, $k \in K_0$ the inequalities are take place

$$|F_l(x, u, p, k)| \le \alpha^{(h)}(x) + M_3 |p|^{r_2^*},$$

$$|G_l(x, u, p)| \le \beta^{(h)}(x) + M_4 |p|^{r_2^*}, \quad (l = \overline{0, l_0}),$$

for $n \ge 4$ there exist such functions $\alpha(x), \beta(x) \in L_1(\Omega)$ and constants $M_5, M_6 > 0$, that for almost all $x \in \Omega$ and for all $u \in R$, $p \in E_n$, $k \in K_0$ the inequalities are take place

$$|F_l(x, u, p, k)| \le \alpha(x) + M_5(|u|^{r_1^*} + |p|^{r_2^*}),$$

$$|G_l(x, u, p)| \le \beta(x) + M_6(|u|^{r_1^*} + |p|^{r_2^*}), \quad (l = \overline{0, l_0}),$$

where r_1^*, r_2^* are some numbers that satisfy the following conditions

$$r_1^* \in [2, \infty) \text{ at } n = 4, r_1^* \in [2, 2n/(n - 4)) \text{ for } n \ge 5, \tag{2.6}$$

$$r_2^* \in [2, \infty) \text{ at } n = 2, r_2^* \in [2, 2n/(n - 2)) \text{ for } n \ge 3, \tag{2.7}$$

5) set of admissible controls, satisfying constraints (2.4) is not empty, i.e. $W = \{v(x) \in V_{ad} : J_l(v) \le 0(l = \overline{1, l_0})\} \ne \emptyset$.

Under a solution of the boundary value problem (2.1), (2.2) corresponding to the controling function $v(x) \in V_{ad}$, we will understand the generalized solution from $\overset{0}{W}{}_2^1(\Omega)$ of this boundary problem, i.e. function $u = u(x, v)$ from $\overset{0}{W}{}_2^1(\Omega)$ satisfyed the following integral identity

$$\int_{\Omega}\left[\sum_{i,j=1}^{n} a_{ij}(x,k(x))u_{x_j}\eta_{x_i}+q(x)a(x,u)\eta\right]dx=\int_{\Omega} f(x)\eta dx \quad (2.8)$$

for all $\eta=\eta(x)$ fray $\overset{0}{W}{}_2^1(\Omega)$.

Using the theory of monotone operators [17, p. 94], and also results from [16, pp. 354-368], easy to verife follows that for each fixed $v(x)\in V_{ad}$ an unique generalized solution $u(x,v)$ of the problem (2.1), (2.2) exists from spacc $W_2^1(\Omega)$. Moreover,this in generalized solution $u(x,v)$ of the problem (2.1), (2.2) belongs to the space $W_{2,0}^2(\Omega)=W_2^2(\Omega)\cap \overset{0}{W}{}_2^1(\Omega)$ also, satisfies the equation (2.1) at almost all $x\in\Omega$ and following a priory estimation is takes place

$$\|u\|_{2,\Omega}^{(2)}\le M_1\,\|f\|_{2,\Omega}\,. \quad (2.9)$$

It is well known [18, p.78] that enclosures $W_2^2(\Omega)\to L_{r_1}(\Omega)$, $\quad W_2^1(\Omega)\to L_{r_2}(\Omega)$ are bounded, if the numbers r_1 and r_2 are satisfy the conditions:

$$r_1=\infty \text{ for } n=2 \text{ or } n=3,\quad r_1\ge 2 \text{ for } n=4,\quad r_1=2n/(n-4) \text{ for } n\ge 5, \quad (2.10)$$

$$r_2\ge 2\ \text{ for } n=2,\quad r_2=2n/(n-2)\ \text{ for } n\ge 3. \quad (2.11)$$

From the inequality (2.9), it follows that the following estimation is take plase:

$$\|u\|_{r_1,\Omega}+\|u_x\|_{r_2,\Omega}\le M_2\,\|f\|_{2,\Omega}\,. \quad (2.12)$$

Moreover, from the condition 4) it is follows that the operators generating by following functions

$$F_l(x,u(x,v),u_x(x,v),k(x)),\qquad G_l\ (x,u(x,v),u_x(x,v))\qquad (l=\overline{0,l_0})$$

are operate from $L_{r_1^*}(\Omega)\times L_{r_2^*}(\Omega)\times K$, $L_{r_1^*}(\Omega)\times L_{r_2^*}(\Omega)$ to $L_1(\Omega)$, $L_1(\Omega)$ accordingly [19, p. 376]. From this it follows that the functional $J_0(v)$ is defined in W and takes finite value.

3 Correctness of the problem

Let us introduce the space $B=(W_{s_1}^1(\Omega))^r\times L_{s_2}(\Omega)$, where $s_1>n$ for $n\ge 2, s_2=2$ at $n=2$ and $n=3$, $\quad s_2>n/2$ for all $n\ge 4$.

Theorem 3.1. *Let the conditions 1)-5) are satisfied. Then of for problem (2.1)-(2.5) there is exest at least one optimal control $v_*(x)=(k_*(x),q_*(x))\in V_{ad}$ i.e.*

$$J_{0*}\ =\inf\{J_0(v):v=v(x)\in V_{ad}\}>-\infty, V_*=\{v_*(x)\in V_{ad}:J_0(v_*)=J_{0*}\}\ne\emptyset.$$

Set of optimal controls V_* of the problem (2.1)-(2.5) is weakly compact on B and arbitrary minimizing sequence $\{v^{(m)}(x)\}=\{(k^{(m)}(x),q^{(m)}(x))\}\subset V_{ad}$ of the functional $J_0(v)$ converges weakly to the set V_* in B.

Let us show that the functional $J_0(v)$ is weakly continuous on the set V_{ad} in B. Let $v(x) = (k(x), q(x)) \in V_{ad}$ be some element and $\{v^{(m)}(x)\} = \{(k^{(m)}(x), q^{(m)}(x))\} \subset V_{ad}$ be an arbitrary sequence converging weakly to the element $v(x)$ in B, i.e.

$$k^{(m)}(x) \to k(x) \text{ weak in } (W^1_{s_1}(\Omega))^r, \tag{3.1}$$

$$q^{(m)}(x) \to q(x) \text{ weak in } L_{s_2}(\Omega). \tag{3.2}$$

From compactness of the enclosure $(W^1_{s_1}(\Omega))^r \to (L_\infty(\Omega))^r$ [18, p. 78] and from (3.1) it follows that

$$k^{(m)}(x) \to k(x) \text{ strong in } (L_\infty(\Omega))^r. \tag{3.3}$$

Besides, owing to single-valued solvability of the problem (2.1), (2.2), for each control $v^{(m)}(x) \in V_{ad}$ it corresponds a unique solution $u^{(m)}(x) = u(x, v^{(m)})$ of the problem (2.1), (2.2) and the following estimate is valid:

$$\left\|u^{(m)}\right\|^{(2)}_{2,\Omega} \le M_7 \quad (m = 1, 2, ...). \tag{3.4}$$

Then from compactness of the enclosures $W^2_2(\Omega) \to W^1_2(\Omega)$, $W^1_2(\Omega) \to L_{r_1^*}(\Omega)$, $W^1_2(\Omega) \to L_{r_2^*}(\Omega)$, [18, p. 78] follows that from the sequence $\{u^{(m)}\}$ it is possible to extract subsequence $\{u^{(m_k)}\}$ such that

$$u^{(m_k)}(x) \to u(x) \text{ weak in } W^2_2(\Omega) \text{ and strong in } W^1_2(\Omega), L_{r_1^*}(\Omega), \tag{3.5}$$

$$u^{(m_k)}_{x_i}(x) \to u_{x_i}(x)(i = \overline{1, n}) \text{ strong in } L_{r_2^*}(\Omega), \tag{3.6}$$

where $u(x) \in W^2_{2,0}(\Omega)$ is some element, $r_1^* = \infty$ at $n = 2$ and $n = 3$, the number r_1^* for $n \ge 4$, and the number r_2^* for $n \ge 2$ satisfy conditions (2.6), (2.7).

Now, we like show that $u(x)$is a solution to the problem (2.1), (2.2), corresponding to the control $v(x) \in V_{ad}$, i.e. $u(x) = u(x, v)$. It is clear that the following identities are valid:

$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x, k^{(m_k)}(x)) u^{(m_k)}_{x_i} \eta_{x_j} + q^{(m_k)}(x) a(x, u^{(m_k)}) \eta\right] dx = \int_\Omega f(x) \eta \, dx, \ (k = 1, 2, ...),$$
$$\forall \eta = \eta(x) \in \overset{0}{W}{}^1_2(\Omega). \tag{3.7}$$

On the basis of relations (3.3)-(3.6) and constraints on the functions $a_{ij}(x, k)$, $(i, j = \overline{1, n})$ we obtain

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^n a_{ij}(x, k^{(m_k)}(x)) u^{(m_k)}_{x_i} \eta_{x_j} dx = \int_\Omega \sum_{i,j=1}^n a_{ij}(x, k^{(m_k)}(x))(u^{(m_k)}_{x_i} - u_{x_i}) \eta_{x_j} dx + \\ &+ \int_\Omega \sum_{i,j=1}^n a_{ij}(x, k^{(m_k)}(x)) u_{x_i} \eta_{x_j} dx \to \int_\Omega \sum_{i,j=1}^n a_{ij}(x, k(x)) u_{x_i} \eta_{x_j} dx \end{aligned} \tag{3.8}$$

Besides, it is easy to see that

$$\int\limits_{\Omega} q^{(m_k)}(x)a(x,u^{(m_k)})\eta\,dx = \int\limits_{\Omega} q^{(m_k)}(x)[a(x,u^{(m_k)}) - a(x,u)]\eta dx + \int\limits_{\Omega} q^{(m_k)}(x)a(x,u)\eta\,dx. \tag{3.9}$$

Using the condition 1), the Cauchy-Bunyakovsky inequality, the inequalities $0 \le q_0 \le q^{(m_k)}(x) \le q_1$ $(k = 1, 2, ...)$ a.e. in Ω and the relation (3.5), we obtain

$$\left| \int\limits_{\Omega} q^{(m_k)}(x)[a(x,u^{(m_k)}) - a(x,u)]\eta dx \right| \le q_1 L \left\| u^{(m_k)} - u \right\|_{2,\Omega} \|\eta\|_{2,\Omega} \to 0. \tag{3.10}$$

Further, using the inclusions $u(x) \in L_{r_1}(\Omega)$, $\eta(x) \in L_{r_2}(\Omega)$, where the numbers r_1, r_2 satisfy conditions (2.10), (2.11) and imposed conditions on number s_2, it is easy to verify that $a(x,u)\eta \in L_{s_2/(s_2-1)}(\Omega)$. Then from convergence (3.2) it is received that

$$\int\limits_{\Omega} q^{(m_k)}(x)a(x,u)\eta dx \to \int\limits_{\Omega} q(x)a(x,u)\eta dx. \tag{3.11}$$

Then passing to the limit in the equality (3.9) and considering (3.10), (3.11), we obtain

$$\int\limits_{\Omega} q^{(m_k)}(x)a(x,u^{(m_k)})\eta dx \to \int\limits_{\Omega} q(x)a(x,u)\eta dx$$

At last, passing to the limit in the equality (3.7) and considering (3.8), (3.11) we obtain that $u(x)$ satisfies the identity (2.8), i.e. is the generalized solution to the problem (2.1), (2.2), from $W_2^1(\Omega)$, corresponding to the control $v(x) \in V_{ad}$. From this and from the inclusion $u(x) \in W_{2,0}^2(\Omega)$ follow that $u(x) = u(x,v)$.

Thus, it is set up that satisfying relations (3.1), (3.2) it is possible to select the subsequence $\{u^{(m_k)}\}$ from sequence $\{u^{(m)}\}$ for which relations (3.5), (3.6) are valid, where $u(x) = u(x,v)$. It is easy to verify that relations (3.5), (3.6), are valid not only for the subsequence $\{u^{(m_k)}\}$ but also for all sequences $\{u^{(m)}\}$, i.e.

$$u^{(m)}(x) \to u(x,v) \text{weak in } W_2^2(\Omega) \text{ and strong in } W_2^1(\Omega), L_{r_1^*}(\Omega), \tag{3.12}$$

$$u_{x_i}^{(m)}(x) \to u_{x_i}(x,v) \quad (i = \overline{1,n}) \text{ strong in } L_{r_2^*}(\Omega). \tag{3.13}$$

Besides, from the condition 4) it follows that the operators generated by functions

$$F_l(x,u(x,v),u_x(x,v),k(x)), \quad G_l\ (x,u(x,v),u_x(x,v)) \quad (l = \overline{0,l_0})\ ,$$

continuously operate from $L_{r_1^*}(\Omega) \times L_{r_2^*}(\Omega) \times K$, $L_{r_1^*}(\Omega) \times L_{r_2^*}(\Omega)$ in $L_1(\Omega), L_1(\Omega)$ accordingly [see:19, p. 376]. Then, using relations (3.2), (3.3), (3.12) and (3.13) we obtain that $J_0(v^{(m)}) \to J_0(v)$ at $m \to \infty$. It means that the functional $J_0(v)$ is weak in B, is continuous on the set V_{ad}, and on the set W also.

Show that set W is weakly compact in the space B. Let

$$\{v^{(m)}(x)\} = \{k^{(m)}(x), q^{(m)}(x)\} \subset W$$

be an arbitrary sequence, i.e.

$$v^{(m)}(x) = (k^{(m)}(x), q^{(m)}(x)) \in V, \quad J_l(v^{(m)}) \leq 0 \quad (l = \overline{1, l_0}; \quad m = 1, 2, ...).$$

The set V_{ad} is convex, closed and bounded in a reflexive Banach space B [20, p. 51]. Then from the sequence $\{v^{(m)}(x)\} = \{(k^{(m)}(x), q^{(m)}(x))\} \subset V_{ad}$, it is possible to select the subsequence $\{v^{(m_k)}(x)\} = \{(k^{(m_k)}(x), q^{(m_k)}(x))\} \subset V_{ad}$ such that $k^{(m_k)}(x) \to k(x)$weak in $(W^1_{s_1}(\Omega))^r$, $q^{(m_k)}(x) \to q(x)$ weak in $L_{s_2}(\Omega)$, where $v(x) = (k(x), q(x)) \in V_{ad}$ is some element. Repeating the reasoning above at the proof of a weakly continuity of the functional $J_0(v)$, and passing to the limit in the inequalities $J_l(v^{(m_k)}) \leq 0 \quad (l = \overline{1, l_0})$, we obtain thoot $J_l(v) \leq 0 \quad (l = \overline{1, l_0})$. It means that $v(x) \in W$, i.e. the set W is a weak compact in B. Then, applying results from [20, p. 49], we set up that the problem (2.1)-(2.5) exist. The proof of theorem 1 is complete.

Remark 1. The problem considering in the [13] is particyler case of the problem (2.1)-(2.5. Then from examples given in the work [13] show that a solution of the problem (2.1)-(2. 5) can be non-unique and minimizing sequence for the functional $J_0(v)$ can not have limit in the space B, i.e. the problem (2.1)-(2.5) is incorrect in the metric of space B.

4 Necessary condition of optimality

Let following conditions are satisfied:

6) the functions $a_{ij}(x,k)(i,j = \overline{1,n})$, $a(x,u)$, $F_l(x,u,p,k)$, $G_l(x,u,p)(l = \overline{0,l_0})$ have partial derivatives $a_{ijk_m}(x,k)(i,j = \overline{1,n};\ m = \overline{1,r})$, $a_u(x,u)$, $F_{lu}(x,u,p,k)$, $F_{lp_i}(x,u,p,k)$, $F_{lk_m}(x,u,p,k)$, $G_{lu}(x,u,p)$, $G_{lp_i}(x,u,p)(l = \overline{0,l_0};\ i,j = \overline{1,n};\ m = \overline{1,r})$ that are measurable with respect to $x \in \Omega$, and are continuous with respect to $x \in \Omega,\ u \in R,\ p \in E_n,\ k \in K_0$;

7) $a_u(x,u) \geq 0$ at almost all $x \in \Omega$ and for all $u \in R$; the operators generated by functions $a_{ijk_m}(x,k(x))(i,j = \overline{1,n};\ m = \overline{1,r})$, $a_u(x,u(x))$, $F_{lu}(x,u(x),u_x(x),k(x))$, $F_{lp_i}(x,u(x),u_x(x),k(x))$, $F_{lk_m}(x,u(x),u_x(x),k(x))$, $G_{lu}(x,u(x),u_x(x))$, $G_{lp_i}(x,u(x),u_x(x))$ $(l = \overline{0,l_0};\ i,j = \overline{1,n};\ m = \overline{1,r})$ continuously operate from $(W^1_\infty(\Omega))^r$, $L_{r_1}(\Omega)$, $L_{r_1}(\Omega) \times (L_{r_2}(\Omega))^n \times (W^1_\infty(\Omega))^r$, $L_{r_1}(\Omega) \times L_{r_2}(\Omega) \times (W^1_\infty(\Omega))^r$, $L_{r_1}(\Omega) \times (L_{r_2}(\Omega))^n \times W^1_\infty(\Omega)$, $L_{r_1}(\Omega) \times (L_{r_2}(\Omega))^n$, $L_{r_1}(\Omega) \times (L_{r_2}(\Omega))^n$ in $L_\infty(\Omega)$, $L_s(\Omega)$, $L_2(\Omega)$, $L_2(\Omega)$, $L_1(\Omega)$, $L_2(\Omega)$, $L_2(\Omega)$ accordingly, where $s = 2$, at $n = 2$ and $n = 3$, $s > \frac{n}{2}$ for $n \geq 4$.

For the problem (2.1)-(2.5) we introduce conjugate states $\psi_l = \psi_l(x,v)$, $l = \overline{0,l_0}$ are the solutions of the following problems

$$-\sum_{i,j=1}^{n} \left(a_{ij}(x,k(x))\,\psi_{lx_i}\right)_{x_j} + q(x)\,a_u(x,u)\,\psi_l =$$

$$= -F_{lu} - q(x)\, G_{lu} + \sum_{i=1}^{n} \left(F_{lp_i} + q(x)\, G_{lp_i}\right)_{x_i}\ ,\ x \in \Omega, \tag{4.1}$$

$$\psi_l(x, v) = 0, \quad x \in \Gamma \quad \left(l = \overline{0, l_0}\right), \tag{4.2}$$

where $u = u(x, v)$ is a solution of the problem (2.1),(2.2). Under a solution of the problems (4.1), (4.2), at each fixed control $v \in V_{ad}$, we are understand a generalized solution from $W_2^1(\Omega)$, i.e. the functions $\psi_l = \psi_l(x, v)\ l = \overline{0, l_0}$ from $\overset{0}{W_2^1}(\Omega)$ satisfying the integral identites

$$\begin{aligned}
&\int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(x, k(x))\, \psi_{lx_i} \eta_{x_j} + q(x)\, a_u(x, u)\, \psi_l \eta\right] dx = \\
&= -\int_\Omega \left[\left(F_{lu} + q(x)\, G_{lu}\right) \eta + \sum_{i=1}^{n} \left(F_{lp_i} + q(x)\, G_{lp_i}\right) \eta_{x_i}\right] dx \\
&\left(l = \overline{0, l_0}\right), \quad \forall \eta = \eta(x) \in \overset{0}{W_2^1}(\Omega)
\end{aligned} \tag{4.3}$$

From results of the book [16, pp. 181-200] it follows that in conditions of the problem (4.1), (4.2) has an unique generalized solution from $W_2^1(\Omega)$ for each given $v(x) \in V_{ad}$ and the following estimates are satisfied

$$\|\psi_l\|_{2,\Omega}^{(1)} \le M_8 \left[\|F_{lu}\|_{2,\Omega} + \|G_{lu}\|_{2,\Omega} + \left\|\sqrt{\sum_{i=1}^{n} F_{lp_i}^2}\right\|_{2,\Omega} + \left\|\sqrt{\sum_{i=1}^{n} G_{lp_i}^2}\right\|_{2,\Omega}\right] \quad \left(l = \overline{0, l_0}\right) \tag{4.4}$$

The enclosure $W_2^1(\Omega) \to L_{r_2}(\Omega)$ is bounder [18, p. 78] . Using this estimates, we obtain that

$$\|\psi_l\|_{r_2,\Omega} \le M_9 \left[\|F_{lu}\|_{2,\Omega} + \|G_{lu}\|_{2,\Omega} + \left\|\sqrt{\sum_{i=1}^{n} F_{lp_i}^2}\right\|_{2,\Omega} + \left\|\sqrt{\sum_{i=1}^{n} G_{lp_i}^2}\right\|_{2,\Omega}\right] \quad \left(l = \overline{0, l_0}\right), \tag{4.5}$$

in which the number r_2 satisfies the condition (2.10).

Theorem 4.1. *Let conditions 1)-7) are satisfied, $v_*(x) = (k_*(x), q_*(x)) \in V_{ad}$ is optimal control for the problem (2.1)-(2.5), $u_*(x) = u(x, v_*)$, $\psi_{l*}(x) = \psi_l(x, v_*)$ $\left(l = \overline{0, l_0}\right)$, are solutions of the problems (2.1), (2.2) and (4.1), (4.2), corresponding to the control $v_*(x)$. Then there are numbers $\lambda_l^* \ge 0$ $\left(l = \overline{0, l_0}\right)$ simultaneously unequal to zero such that at almost all $x \in \Omega$ and for any $k(x) \in K$, $q \in [q_0, q_1]$ the following inequality is satisfied*

$$\sum_{l=0}^{l_0} \lambda_l^* \left\{ \int_\Omega \sum_{m=1}^{r} \left[\sum_{i,j=1}^{n} a_{ijk_m}(x, k_*(x)) u_{*x_j} \psi_{l*x_i} + F_{lk_m}(x, u_*(x), u_{*x}(x), k_*(x))\right] \times \right.$$

$$[k_m(x)-k_{*m}(x)]\,dx+[a\left(x,u_*(x)\right)\psi_{l*}(x)+G_l\left(x,u_*(x),u_{*x}(x)\right)][q-q_*(x)]\}\geq 0 \tag{4.6}$$

Let us introduce a variation of control functions $k_*\left(x\right)$ and $q_*\left(x\right)$. For the function $k_*\left(x\right)$ we define a classic variation

$$k_\varepsilon\left(x\right)=k_*\left(x\right)+\varepsilon^n\left[k\left(x\right)-k_*\left(x\right)\right],\quad x\in\Omega,$$

where $\varepsilon\in(0,1)$ is any number, $k_*\left(x\right)\in K$. From convex ness of the set K in $\left(W_\infty^1\left(\Omega\right)\right)^r$ it follows that $k_\varepsilon\left(x\right)\in K$ for all $\varepsilon\in(0,1)$. It is obvious that

$$k_\varepsilon\left(x\right)\to k_*\left(x\right)\ \text{strong in}\ (W_\infty^1(\Omega))^r\ \text{for}\ \varepsilon\to 0. \tag{4.7}$$

For the function $q_*\left(x\right)$ we construct a multipoint impulse variation $q_\varepsilon\left(x\right)$. We take a finite set pairwise various points of Lebesgue $x^i\in\Omega\left(i=\overline{1,N}\right)$ for all considered functions. Let $\beta_i^m(i=\overline{1,N};\quad m=\overline{1,M})$ be any real numbers. We define the parallelepipeds

$$\Pi_{ik}^\varepsilon=\Big\{x=(x_1,\ldots,x_n)\in\Omega:\ x_1^i-\varepsilon\sum_{l=1}^k\beta_i^l\leq x_1<x_1^i-\varepsilon\sum_{l=1}^{k-1}\beta_i^l,$$

$$x_s^i-\varepsilon k\leq x_s<x_s^i-\varepsilon(k-1)\ (s=\overline{2,n})\Big\}$$

$$\left(i=1,N;\ \ k=\overline{1,M}\right).$$

For sufficiently small $\varepsilon>0$ parallelepipeds Π_{ik}^ε are not intersected, and the volume Π_{ik}^ε is $|\Pi_{ik}^\varepsilon|=\beta_i^k\varepsilon^n$. A variation $q_\varepsilon\left(x\right)$ which parameters are sets $\{x^i\}$, $\left\{\beta_i^k\right\}$, $\left(i=\overline{1,N};\ \ k=\overline{1,M}\right)$, we define as follows:

$$q_\varepsilon\left(x\right)=\begin{cases} q_i^k, & x\in\Pi_{ik}^\varepsilon,\\ q_*\left(x\right), & x\in\Omega\backslash\bigcup\limits_{i,k}\Pi_{ik}^\varepsilon,\end{cases} \tag{4.8}$$

where $q_i^k\in[q_0,q_1]\quad\left(i=\overline{1,N};\ \ k=\overline{1,M}\right)$ is any number. It is obvious that $q_\varepsilon\left(x\right)\in Q$ for all sufficiently small $\varepsilon>0$ and

$$q_\varepsilon\left(x\right)\to q\left(x\right)\ \text{strong in}\ L_p(\Omega) \tag{4.9}$$

for $\varepsilon\to 0$, where $p\in[1,\infty)$ is any finite number.

Let us designate $v_\varepsilon\left(x\right)=\left(k_\varepsilon\left(x\right),q_\varepsilon\left(x\right)\right)$, $u_\varepsilon\left(x\right)=u\left(x,v_\varepsilon\right)$, $\Delta u_\varepsilon\left(x\right)=u_\varepsilon\left(x\right)-u_*\left(x\right)$, $\Delta_\varepsilon a_{ij}=a_{ij}\left(x,v_\varepsilon\right)-a_{ij}\left(x,v_*\right)\ \left(i,j=\overline{1,n}\right)$, $\Delta_\varepsilon k_i\left(x\right)=k_{\varepsilon i}\left(x\right)-k_{*i}\left(x\right)\ (i=\overline{1,n})$, $\Delta_\varepsilon q=q_\varepsilon\left(x\right)-q_*\left(x\right)$. From the conditions (2.1), (2.2) it follows that Δu_ε is a linearized solution to the following boundary value problem in $W_{2,0}^2\left(\Omega\right)$

$$-\sum_{i,j=1}^n\left(a_{ij}\left(x,k_\varepsilon\left(x\right)\right)\Delta u_{\varepsilon x_j}\right)_{x_i}+q_\varepsilon\left(x\right)a_u\left(\xi_\varepsilon\right)\Delta u_\varepsilon=$$

$$= \sum_{i,j=1}^{n} \left(\Delta_\varepsilon a_{ij} u_{*x_j}\right)_{x_i} - \Delta_\varepsilon q a\left(x, u_*\right), \quad x \in \Omega, \tag{4.10}$$

$$\Delta u_\varepsilon\left(x\right) = 0, \quad x \in \Gamma, \tag{4.11}$$

where $\xi_\varepsilon = (x, u_* + \theta\Delta u_\varepsilon)$, $\theta \in [0,1]$. For the Δu_ε the following estimate is valid [16, p. 221]

$$\|\Delta u_\varepsilon\|_{2,\Omega}^{(2)} \leq M_{10}\left[\sum_{i,j=1}^{n}\left(\left\|\Delta_\varepsilon a_{ij} u_{*x_j x_i}\right\|_{2,\Omega} + \left\|\left(\Delta_\varepsilon a_{ij}\right)_{x_i} u_{*x_j}\right\|_{2,\Omega}\right) + \left\|\Delta_\varepsilon q a\left(x, u_*\right)\right\|_{2,\Omega}\right], \tag{4.12}$$

Using Holder's inequality, the conditions imposed on numbers s_1 and r_2, estimate (2.11), relation (4.7) and conditions 1) - 6) for the functions $a_{ij}(x,k)$ $(i = \overline{1,n})$, we have

$$\sum_{i,j=1}^{n}\left(\left\|\Delta_\varepsilon a_{ij} u_{*x_j x_i}\right\|_{2,\Omega} + \left\|(\Delta_\varepsilon a_{ij})_{x_i} u_{*x_j}\right\|_{2,\Omega}\right) \leq$$

$$\sum_{i,j=1}^{n}\left(\|\Delta_\varepsilon a_{ij}\|_{00,\Omega}\left\|u_{*x_j x_i}\right\|_{2,\Omega} + \|(\Delta_\varepsilon a_{ij})_{x_i}\|_{s_1,\Omega}\left\|u_{*x_j}\right\|_{2s_1/(s_1-2),\Omega}\right) \leq$$

$$\sum_{i,j=1}^{n}\left(\|\Delta_\varepsilon a_{ij}\|_{00,\Omega}\left\|u_{*x_j x_i}\right\|_{2,\Omega} + \left\|\left(\Delta_\varepsilon a_{ij}\right)_{x_i}\right\|_{s_1,\Omega}\left\|u_{*x_j}\right\|_{r_2,\Omega}\right) \to 0 \tag{4.13}$$

when $\varepsilon \to 0$. Besides, using Holder's inequality, condition 3) for function $a(x,u)$, estimate (2.12) and relation (4.9), we obtain

$$\left\|\Delta_\varepsilon q a\left(x, u_*\right)\right\|_{2,\Omega} \leq \left\|\Delta_\varepsilon q\right\|_{r_3,\Omega}\left\|a\left(x, u_*\right)\right\|_{r_1,\Omega} \leq L\left\|\Delta_\varepsilon q\right\|_{r_3,\Omega}\left\|u_*\right\|_{r_1,\Omega} \to 0 \tag{4.14}$$

when $\varepsilon \to 0$, where the number r_1 is determined by the condition (2.10), $r_3 = 2$ at $n = 2$ and $n = 3$, $r_3 = \frac{2r_1}{r_1-2}$, $r_1 > 2$ at $n = 4$, $r_3 = \frac{n}{2}$ for $n \geq 5$. Then taking into account the relations (4.13) and (4.14) in the inequality (4.12), we obtain the following convergence

$$\|\Delta \mathrm{u}_\varepsilon\|_{2,\Omega}^{(2)} = \|u_\varepsilon - u_*\|_{2,\Omega}^{(2)} \to 0 \text{ when } \varepsilon \to 0 \tag{4.15}$$

Let us calculate the first-order variations of the functionals $J_l(v)$, $(l = \overline{0, l_0})$. Using condition 6), 7) the increment of the functional $J_l(v)$ at the point v_*, it is possible to present as follows:

$$\Delta_\varepsilon J_l(v_*) = J_l(v_\varepsilon) - J_l(v_*) =$$
$$= \int_\Omega [F_{lu}\left(\eta_{\varepsilon l}\right)\Delta u_\varepsilon + \sum_{i=1}^{n} F_{lp_i}\left(\mu_{\varepsilon l}\right)\Delta u_{\varepsilon x_i} + q_\varepsilon(x)(G_{lu}\left(v_{\varepsilon l}\right)\Delta u_\varepsilon + \sum_{i=1}^{n} G_{lp_i}\left(\xi_{\varepsilon l}\right)\Delta u_{\varepsilon x_i})]dx +$$

$$+\varepsilon^n \int\limits_{\Omega} \sum_{m=1}^{r} F_{lk_m}(\overline{\mu}_{\varepsilon l})\,[k_m(x) - k_{*m}(x)]dx + \sum_{i=1}^{N}\sum_{k=1}^{M} \int\limits_{\Pi_{ik}^{\varepsilon}} G_l(x, u_*, u_{*x})\,[q_{ik} - q_*(x)]dx. \tag{4.16}$$

where

$$\eta_{\varepsilon l} = (x, u_* + \theta_{1l}\Delta u_\varepsilon, k_{\varepsilon x}, k_\varepsilon), \quad \mu_{\varepsilon l} = (x, u_*, u_{*x} + \theta_{2l}\Delta u_{\varepsilon x}, k_\varepsilon),$$

$$\nu_{\varepsilon l} = (x, u_* + \theta_{3l}\Delta u_\varepsilon, u_{\varepsilon x}), \quad \xi_{\varepsilon l} = (x, u_*, u_{*x} + \theta_{4l}\Delta u_{\varepsilon x_i}),$$

$$\overline{\mu}_{\varepsilon l} = (x, u_*, u_{*x}, k_* + \theta_{5l}(k_\varepsilon(x) - k_*(x))),$$

$$\theta_{il} \in [0,1] \quad \left(i = \overline{1,5};\ l = \overline{0,l_0}\right)$$

Let $\psi_{l\varepsilon}(x) = \psi_l(x, v_\varepsilon)$ be generalized solution of the following boundary value problem in $W_2^1(\Omega)$

$$-\sum_{i,j=1}^{n} (a_{ij}(x, k_\varepsilon(x))\Psi_{l\varepsilon x_i})_{x_j} + q_\varepsilon(x) a_u(x, u_* + \theta\Delta u_\varepsilon)\,\psi_{l\varepsilon} =$$

$$\sum_{i=1}^{n} \left(F_{lp_i}(\mu_{\varepsilon l}) + q_\varepsilon(x)\, G_{lp_i}(\xi_{\varepsilon l})\right)_{x_i} - \left(F_{lu}(\eta_{\varepsilon l}) + q_\varepsilon(x)\, G_{lu}(v_{\varepsilon l})\right), \quad x \in \Omega \tag{4.17}$$

$$\psi_{\varepsilon l}(x) = 0, \quad x \in \Omega \quad \left(l = \overline{0,l_0}\right) \tag{4.18}$$

where $\theta \in [0,1]$. Under the generalized solution to the boundary value problem (4.17), (4.18) we understand the function $\psi_{\varepsilon l}(x)$ satisfying the following integral identity in $\overset{0}{W}{}_2^1(\Omega)$

$$\int\limits_{\Omega} \left[a_{ij}(x, k_\varepsilon(x))\,\psi_{l\varepsilon x_i}\eta_{x_j} + q_\varepsilon(x)\,\alpha_u(x, u_* + \theta\Delta u_\varepsilon)\,\psi_{l\varepsilon}\eta\right]dx =$$

$$-\int\limits_{\Omega} \left\{[F_{lu}(\eta_{\varepsilon l}) + q_\varepsilon(x)\, G_{lu}(\nu_{\varepsilon l})]\,\eta + + \sum_{i=1}^{n} [F_{lp_i}(\mu_{\varepsilon l}) + q_\varepsilon(x)\, G_{lp_i}(\xi_{\varepsilon l})]\eta_{x_i}\right\} dx,$$

$$\forall \eta = \eta(x) \in \overset{0}{W}{}_2^1(\Omega),\ l = \overline{0,l_0} \tag{4.19}$$

Using relations (4.4), (4.5), (4.7), (4.9), (4.15) and condition 6),7) for the solution to the problem (4.17), (4.18) it is possible to show that, i.e.

$$\|\Delta\psi_{l\varepsilon}\|_{2,\Omega}^{(1)} = \|\psi_{l\varepsilon} - \psi_*\|_{2,\Omega}^{(1)} \to 0 \quad (l = \overline{0,l_0}) \text{ as } \varepsilon \to 0. \tag{4.20}$$

It is clear that a solution to the boundary value problem (4.10), (4.11) satisfies the following integral identity

$$\begin{array}{l}\int\limits_{\Omega}\left[\sum\limits_{i,j=1}^{n} a_{ij}\left(x,k_{\varepsilon}\left(x\right)\right)\Delta u_{\varepsilon x_j}\eta_{x_i}+q_{\varepsilon}\left(x\right)a_u\left(x,u_*+\theta\Delta u_{\varepsilon}\right)\Delta u_{\varepsilon}\eta\right]dx=\\ -\int\limits_{\Omega}\left[\sum\limits_{i,j=1}^{n}\Delta_{\varepsilon}a_{ij}u_{*x_j}\eta_{x_i}+\Delta_{\varepsilon}qa\left(x,u_*\right)\eta\right]dx\qquad\forall\eta=\eta\left(x\right)\in\overset{\circ}{W}{}_2^1\left(\Omega\right)\end{array},\tag{4.21}$$

Taking in (4.19) $\eta=\Delta \mathrm{u}_{\varepsilon}$, and in (4.21) $\eta=\psi_{l\varepsilon}$, subtracting these equalities and taking into account the obtaining equality (4.16), we have

$$\begin{array}{l}\Delta_{\varepsilon}J_l\left(v_*\right)=\int\limits_{\Omega}\left[\sum\limits_{i,j=1}^{n}\Delta_{\varepsilon}a_{ij}u_{*x_j}\psi_{l\varepsilon x_i}+\Delta_{\varepsilon}qa\left(x,u_*\right)\psi_{l\varepsilon}\right]dx+\\ +\varepsilon^n\int\limits_{\Omega}\left[\sum\limits_{m=1}^{r}F_{lk_m}\left(\overline{\mu}_{\varepsilon l}\right)\left[k_m\left(x\right)-k_{*m}\left(x\right)\right]\right]dx+\\ +\sum\limits_{i=1}^{N}\sum\limits_{k=1}^{M}\int\limits_{\Pi_{ik}^{\varepsilon}}G_l\left(x,u_*,u_{*x}\right)\left[q_{ik}-q_*\left(x\right)\right]dx\quad\left(l=\overline{0,l_0}\right).\end{array}\tag{4.22}$$

Using condition 6), 7) for the functions $a_{ij}\left(x,k\right)\quad\left(i,j=\overline{1,n}\right)$, we have

$$\Delta_{\varepsilon}a_{ij}=\varepsilon^n\sum_{m=1}^{r}a_{ijk_m}\left(x,u_*+\overline{\theta}_{ij}\left(k_{\varepsilon}-k_*\right)\right)\left[k_m\left(x\right)-k_{*m}\left(x\right)\right],\quad\overline{\theta}_{ij}\in[0,1]\quad\left(i,j=\overline{1,n}\right),$$

$$\int\limits_{\Omega}\Delta_{\varepsilon}qa\left(x,u_*\right)\psi_{l\varepsilon}dx=\sum_{i=1}^{N}\sum_{k=1}^{M}\int\limits_{\Pi_{ik}^{\varepsilon}}a\left(x,u_*\right)\psi_{l\varepsilon}\left(x\right)\left[q_{ik}-q_*\left(x\right)\right]dx\quad\left(l=\overline{0,l_0}\right).$$

Taking into account of these relations, the equality (4.22) we can write as follows:

$$\Delta_{\varepsilon}J_l\left(v_*\right)=\Delta_{\varepsilon}^{(1)}J_l\left(v_*\right)+\Delta_{\varepsilon}^{(2)}J_l\left(v_*\right)\quad\left(l=\overline{0,l_0}\right)\tag{4.23}$$

where

$$\Delta_{\varepsilon}^{(1)}J_l\left(v_*\right)=\varepsilon^n\int\limits_{\Omega}\sum_{m=1}^{r}\left[\sum_{i,j=1}^{n}a_{ijk_m}\left(x,k_*+\overline{\theta}_{ij}\left(k_{\varepsilon}-k_*\right)\right)u_{*x_j}\psi_{l\varepsilon x_i}+F_{lk_m}\left(\overline{\mu}_{\varepsilon l}\right)\right]\times$$
$$[k_m(x)-k_{*m}(x)]\,dx,$$

$$\Delta_{\varepsilon}^{(2)}J_l\left(v_*\right)=\sum_{i=1}^{N}\sum_{k=1}^{M}\int\limits_{\Pi_{ik}^{\varepsilon}}\left[a\left(x,u_*\right)\psi_{l\varepsilon}+G_l\left(x,u_*,u_{*x}\right)\right]\left[q_{ik}-q_*\left(x\right)\right]dx.\tag{4.24}$$

The firstorder variation of the functional $J_l(v)$ at the element v_*, determined as follows

$$\delta J_l(v_*) = \lim_{\varepsilon\to 0}\left(\frac{\Delta_\varepsilon^{(1)} J_l(v_*)}{\varepsilon^n} + \frac{\Delta_\varepsilon^{(2)} J_l(v_*)}{\varepsilon^n}\right) = \delta^{(1)} J_l(v_*) + \delta^{(2)} J_l(v_*) \quad \left(l = \overline{0, l_0}\right). \tag{4.25}$$

Now, we show that

$$\delta^{(1)} J_l(v_*) = \int_\Omega \sum_{m=1}^r \left[\sum_{i,j=1}^n a_{ijk_m}(x, k_*(x)) u_{*x_j}\psi_{l*x_i} + F_{lk_m}(x, u_*(x), u_{*x}(x), k_*(x))\right] *$$

$$* \left[k_m(x) - k_{*m}(x)\right] dx, \tag{4.26}$$

$$\delta^{(2)} J_l(v_*) =$$

$$= \sum_{i=1}^N \sum_{k=1}^M \beta_i^k \left[a\left(x^i, u_*\left(x^i\right)\right)\psi_{l*}\left(x^i\right) + G_l\left(x^i, u_*\left(x^i\right), u_{*x}\left(x^i\right)\right)\right]\left[q_{ik} - q_*\left(x^i\right)\right]$$

$$\left(l = \overline{0, l_0}\right) \tag{4.27}$$

Using the equalities (4.24), (4.26), the Cauchy-Bunyakovsky inequality, relation (4.20), and condition 6),7) we have

$$\left|\frac{\Delta_\varepsilon^{(1)} J_l(v_*)}{\varepsilon^n} - \delta^{(1)} J_l(v_*)\right| =$$

$$\left|\int_\Omega \sum_{m=1}^r \left[\sum_{i,j=1}^n a_{ijk_m}\left(x, k_* + \overline{\theta}_{ij}(k_\varepsilon - k_*)\right) u_{*x_j}\psi_{l\varepsilon x_i} + F_{lk_m}(\overline{\mu}_{\varepsilon l})\right]\left[k_m(x) - k_{*m}(x)\right] dx - \right.$$

$$\int_\Omega \sum_{m=1}^r \left[\sum_{i,j=1}^n a_{ijk_m}(x, k_*(x)) u_{*x_j}\psi_{l*x_i} + F_{lk_m}(x, u_*(x), u_{*x}(x), k_*(x))\right] \times$$

$$\left[k_m(x) - k_{*m}(x)\right] dx \le$$

$$\sum_{m=1}^r \left[\sum_{i,j=1}^n \left\|a_{ijk_m}\left(x, k_* + \overline{\theta}_{ij}(k_\varepsilon - k_*)\right) - a_{ijk_m}(x, k_*(x))\right\|_{\infty,\Omega} \left\|u_{*x_j}\right\|_{2,\Omega} \left\|\psi_{l\varepsilon x_i}\right\|_{2,\Omega} + \right.$$

$$\sum_{i,j=1}^n \left\|a_{ijk_m}(x, k_*(x))\right\|_{00,\Omega} \left\|u_{*x_j}\right\|_{2,\Omega} \left\|\Delta\psi_{l\varepsilon x_i}\right\|_{2,\Omega} +$$

$$\|F_{lk_m}(\overline{\mu}_{\varepsilon l}) - F_{lk_m}(x, u_*(x), u_{*x}(x), k_*(x))\|_{1,\Omega}] \|k_m - k_*\|_{00,\Omega} \to 0 \quad (l = \overline{0, l_0})$$

when $\varepsilon \to 0$. From this fact follows the validity of equality (4.26) .

Now, we prove the validity of equality (4.27). For this purpose we set up the following convergence

$$\int_{\Omega_i} \left| \frac{\Delta_\varepsilon^{(2)} J_l(v_*)}{\varepsilon^n} - \delta^{(2)} J_l(v_*) \right| dx \to 0 \quad \left(l = \overline{0, l_0}\right) \tag{4.28}$$

when $\varepsilon \to 0$, where $\Omega_i \subset \Omega$ is some neighborhood of the Lebesgue point $x^i \in \Omega$. Using equalities (4.24), (4.27) and definition of the Lebesgue point, we have

$$\int_{\Omega_i} \left| \frac{\Delta_\varepsilon^{(2)} J_l(v_*)}{\varepsilon^n} - \delta^{(2)} J_l(v_*) \right| dx = \int_{\Omega_i} \left| \frac{1}{\varepsilon^n} \sum_{i=1}^{N} \sum_{k=1}^{M} \int_{\Pi_{ik}^\varepsilon} [a(x, u_*) \psi_{l\varepsilon} + G_l(x, u_*, u_{*x})] \times \right.$$

$$\times [q_{ik} - q_*(x)]\, dx -$$

$$\left. - \sum_{i=1}^{N} \sum_{k=1}^{M} \beta_i^k \left[a\left(x^i, u_*\left(x^i\right)\right) \psi_{l*}\left(x^i\right) + G_l\left(x^i, u_*\left(x^i\right), u_{*x}\left(x^i\right)\right)\right] \left[q_{ik} - q_*\left(x^i\right)\right] \right| dx \leq$$

$$\leq \int_{\Omega_i} \left| \frac{1}{\varepsilon^n} \sum_{i=1}^{N} \sum_{k=1}^{M} \int_{\Pi_{ik}^\varepsilon} [a(x, u_*) \psi_{l*} + G_l(x, u_*, u_{*x})] [q_{ik} - q_*(x)] dx - \right.$$

$$\sum_{i=1}^{N} \sum_{k=1}^{M} \beta_i^k [a(x^i, u_*(x^i)) \psi_{l*}(x^i) + G_l(x^i, u_*(x^i), u_{*x}(x^i))] [q_{ik} - q_*(x^i)] dx +$$

$$\int_{\Omega_i} \left| \frac{1}{\varepsilon^n} \sum_{i=1}^{N} \sum_{k=1}^{M} \int_{\Pi_{ik}^\varepsilon} a(x, u_*) \Delta\psi_{l\varepsilon} [q_{ik} - q_*(x)] dx \right| dx =$$

$$= \frac{o(\varepsilon^n)}{\varepsilon^n} mes\Omega_i + \int_{\Omega_i} \left| \frac{1}{\varepsilon^n} \sum_{i=1}^{N} \sum_{k=1}^{M} \int_{\Pi_{ik}^\varepsilon} a(x, u_*) \Delta\psi_{l\varepsilon} [q_{ik} - q_*(x)] dx \right| dx \quad \left(l = \overline{0, l_0}\right) \tag{4.29}$$

Let us make variable replacement $x = x^i + \xi$ and take designate as follows:

$$\tilde{\Pi}_{ik}^\varepsilon = \left\{ \xi = (\xi_1, \ldots, \xi_n) \in E_n : \ -\frac{\varepsilon}{2} \leq \xi_i < \frac{\varepsilon}{2} \ \left(i = \overline{1, n}\right) \right\}.$$

Then using inequalities $0 \le q_0 \le q_*(x) \le q_1$ a.e. on Ω, $0 \le q_0 \le q_{ik} \le q_1$, $\left(i = \overline{1,N};\quad k = \overline{1,M}\right)$, the Cauchy-Bunyakovsky inequality, estimate (2.9), and relation (4.20), we have

$$\int\limits_{\Omega_i} \left| \frac{1}{\varepsilon^n} \sum_{i=1}^{N} \sum_{k=1}^{M} \int\limits_{\Pi_{ik}^{\varepsilon}} a(x,u_*)\, \Delta\psi_{l\varepsilon}\, [q_{ik} - q_*(x)] dx \right| dx \le$$

$$\frac{1}{\varepsilon^n} \sum_{i=1}^{N} \sum_{k=1}^{M} \int\limits_{\tilde{\Pi}_{ik}^{\varepsilon}} \left\{ \int\limits_{\Omega_i} \left| a\left(x^i+\xi, u_*\left(x^i+\xi\right)\right) \Delta\psi_{l\varepsilon}\left(x^i+\xi\right) \times \left[q_{ik} - q_*\left(x^i+\xi\right)\right] \right| dx \right\} dx \le$$

$$\frac{2q_1}{\varepsilon^n} \sum_{i=1}^{N} \sum_{k=1}^{M} \int\limits_{\tilde{\Pi}_{ik}^{\varepsilon}} \left\{ \int\limits_{\Omega_i} \left| a\left(x^i+\xi, u_*\left(x^i+\xi\right)\right) \right|^2 dx \right\}^{\frac{1}{2}} \left\{ \int\limits_{\Omega_i} \left| \Delta\psi_{l\varepsilon}\left(x^i+\xi\right) \right|^2 dx \right\}^{\frac{1}{2}} dx \to 0$$

when $\varepsilon \to 0$.

From this and (4.29), follows the validity of relation (4.28). It means that the equality (4.27) is satisfied. The set Ω can be covered by the neighborhoods Ω_i. Therefore, it is possible to assert that for almost all $x^i \in \Omega$ the equality (4.27) is satisfied.

Let $v_*(x) = (k_*(x), q_*(x)) \in V_{ad}$ be an optimal control of the problem (2.1)-(2.5). Let us

$$\delta k(x) = k(x) - k_*(x), \quad k(x) \in K.$$

For the function $q_*(x)$ we construct a multipoint impulse variation $q_\varepsilon(x)$. We take a finite set pairwise various points of Lebesgue $x^i \in \Omega$ $\left(i = \overline{1,N}\right)$ all considered functions. Let $\beta_i^k (i = \overline{1,N}; \quad k = \overline{1,M})$ be any real numbers. We define the parallelepipeds

Each set of parameters $\mu = \left(p_i^k, \delta k, q_{ik}, x^i\right)$ corresponds to the variations of functional $A_\mu = \{\delta I_0, \delta I_1, \ldots, \delta I_{l_0}\}$, as a vector in space E_{l_0+1} starting from point $(J_{0*}, 0, ..., 0)$, follows formulas (4.25)-(4.27), where J_{0*} is a minimal value of the functional $J_0(v)$ on the set W. Without losing generality, it is possible to consider that $J_{0*} = 0$. If it is not so then shifting with respect to an axis J_0 we pass to such space in which the vector A_μ goes out from beginning coordinates.

Various set of μ correspond to the set $P \subset E_{l_0+1}$ of functional variation of vectors A_μ. Using linearity of firstorder variation of functional with respect to parameters δ_k, β_i^k, q_{ik}, follows [21] it is ease to show that P is a convex cone in E_{l_0+1}. By enu way we proved that a cone P constructing for optimal control $v_*(x)$ and "negative angle" $L = \left\{A \in E_{l_0+1} : \quad A = (a_0, a_1, \ldots, a_{l_0}),\ a_l \le 0 \quad \left(l = \overline{0, l_0}\right)\ \right\}$ are divided by the nontrivial functional $\lambda^* = \left(\lambda_0^*, \lambda_1^*, \ldots, \lambda_{l_0}^*\right) \in \left(E_{l_0+1}\right)^* = E_{l_0+1}$ in E_{l_0+1}, where $\lambda_l^* \ge 0 \quad \left(l = \overline{0, l_0}\right)$. From this fact and form angle L it follows that

$$\sum_{l=0}^{l_0} \lambda_l^* \delta J_l(v_*) \ge 0. \tag{4.30}$$

Assuming $M = N = \beta_1^1 = 1$, $x^i = x$, $q_{11} = q$, and taking into account relations (4.25)-(4.27) in the expression (4.30), it is convinced of validity of the inequality (4.6).

The proof of Theorem 2 is complete.

Remark 2. It can be shown that under the conditions of Theorem 2 hold the complementary slackness

$$\lambda_l^* J_l(v_*) = 0 \ \left(l = \overline{1, l_0}\right),$$

ie for those indexes which $J_l(v_*) < 0$, may be considered appropriate $\lambda_l^* = 0$, $(l = \overline{1, l_0})$ [21].

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