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ANALYTICAL EXPRESSIONS FOR A SOLUTION OF CONVECTIVE HEAT AND MOISTURE TRANSFER EQUATIONS IN THE FREQUENCY DOMAIN FOR LAYERED MEDIA

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Abstract Analytical expressions for the solution of the differential equations in the frequency domain, describing convective heat and moisture transfer in homogeneous layered medium are obtained. The known connection between systems of differential equations of the second order and the differential matrix Riccati equation is used for the derivation of these expressions. The numerical algorithm is recursive on the number of layers, stable to the rounding errors, and each step of this numerical algorithm based on simple actions: addition, multiplication and inversion of matrices of the second order.

Key words: system of parabolic differential equations, convective heat and moisture transfer, Riccati equation, layer-stripping method.

AMS Mathematics Subject Classification: 35K20

1 Introduction

In this work analytical expressions for the solution of the differential equations in the frequency domain, describing convective heat and moisture transfer in homogeneous layered medium are obtained.

The process of moisture transfer in soils under an influence of a temperature gradient is often observed in the nature. The overalls of the mass transfer are significantly complicated in the presence of a temperature gradient in media due to the additional flow of steam, water and heat. Redistribution of moisture in a soil from the hotter region to the cooler one significantly changes the properties of the soil. Thus, the study of joint heat and moisture transfer is an urgent problem.

We use the well known connection between systems of differential equations of the second order and the differential matrix Riccati equation to obtain the necessary expressions for the layer-stripping method. One of the first layer stripping methods for solving the differential equation of second order for the layered medium is the method proposed in [1]. It has several limitations: the presence of expressions with exponents having indicators with positive real parts, leads to accumulation of rounding errors in a calculated solution. The idea of using the Riccati equation for a construction of the numerical algorithm, which would be convenient for a programming, useful for solving inverse problems and for modeling of electromagnetic wave propagation in horizontally layered media, apparently, was proposed for the first time in the paper [2]. Expressions with exponents having indicators with positive real parts are missing when implementing this algorithm. In this case the layer stripping method does not lead to the accumulation of rounding errors. Further the method was developed in the work [3]. To solve the system of differential equations of the elasticity, the idea of [2] was implemented in the works [4, 5]. The formula for the trace of the solution on the surface $z = 0$ was obtained. It is assumed that the medium is horizontally layered and isotropic. A similar result was obtained in the papers [6, 7] for horizontally-layered isotropic medium with absorption. In the work [8] the horizontally layered transversely isotropic medium with vertical symmetry axis was considered. In the works [4, 5, 6, 7, 8] the authors use the following scheme: the system of differential equations for displacements $\rightarrow$ the system of differential equations for potentials $\rightarrow$ the differential matrix Riccati equation. An intermediate step (the transition to the system for potentials) imposes constraints: first, the transition from the displacements to potentials is ambiguous for some parameters of the problem, secondly, this approach does not have development for media of any kind of the anisotropy. For this reason, the works [9, 10, 11, 12] use the direct transition from the differential system for displacements to the differential matrix Riccati equation. The authors of these works deal with horizontally-layered isotropic (with absorption) media. In the works [13, 14] algorithm for solving the system of differential equations of the elasticity for horizontally layered media of any kind of the anisotropy was proposed. The analytical formulas for the solution are obtained not only on the surface $z = 0$ but also at any point. To study of inverse problems of determining the elastic properties of thinly stratified layers, this algorithm was used in [15, 16, 17, 18]. The layer stripping method for the Maxwell's equations for horizontally layered media with some types of the anisotropy was suggested in [19]. Ideologically, the latter paper duplicates the work [1], for this reason, the authors spend a lot of effort to compute stable. The approach proposed in [13, 14], is expanded on the Maxwell's equations in [20]. It allows to consider media with any type of the anisotropy and to be free from the constraints of [19]. To solve and to study inverse problems of electro-magnetics, this algorithm was used in [21, 22].

The system of differential equations considered in these works, was solved by finite difference method in the papers [23, 24].

2 Problem statement and basic analytical expression

Let media be N-layered structure with boundary points z_k ($k = \overline{0, N}$), $z_0 = 0$, $z_N = H$, and k-th layer be the interval $[z_{k-1}, z_k]$, $h_k = z_k - z_{k-1}$ be the thickness of the layer.

The system of differential equations

$$\begin{aligned} C_0\frac{\partial T}{\partial t} - C_b\frac{\partial \Omega}{\partial t} &= \frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right), \\ \frac{\partial \Omega}{\partial t} &= \frac{\partial}{\partial z}\left(\eta\frac{\partial \Omega}{\partial z} + \mu\frac{\partial T}{\partial z}\right), \end{aligned} \tag{1}$$

describes convective heat and moisture transfer in a homogeneous medium [25, 26, 27]. Here T is a temperature, Ω is a moisture, λ is the coefficient of thermal conductivity, $C_0 = \gamma_0 C$, where C is the coefficient of heat capacity, γ_0 is the specific gravity of a soil, C_b is the coefficient of convective heat transfer, η is the moisture diffusion coefficient of a soil, μ is the thermal transfer coefficient of a soil.

In each interval $[z_{k-1}, z_k]$ the functions λ, C_0, C_b, η, and μ are constant, i.e., these functions are piecewise constant on $z \in [0, H]$ with the values λ_k, $C_{0,k}$, $C_{b,k}$, η_k, and μ_k in k-th layer.

Let the initial conditions

$$T|_{t=0} = T_0(z), \quad \Omega|_{t=0} = \Omega_0(z), \tag{2}$$

the boundary conditions

$$\left(\lambda\frac{\partial T}{\partial z} + \alpha\big(T - T_a\big)\right)\bigg|_{z=0} = 0, \quad T|_{z=H} = T_H, \tag{3}$$

$$\left(\eta\frac{\partial \Omega}{\partial z} + \beta\big(\Omega - \Omega_a\big)\right)\bigg|_{z=0} = 0, \quad \Omega|_{z=H} = \Omega_H \tag{4}$$

be known. Also we assume that the following gluing conditions at points of discontinuity of the medium hold

$$\left[\lambda\frac{\partial T}{\partial z}\right]_{z_k} = 0, \quad [T]_{z_k} = 0, \tag{5}$$

$$\left[\eta\frac{\partial \Omega}{\partial z} + \mu\frac{\partial T}{\partial z}\right]_{z_k} = 0, \quad [\Omega]_{z_k} = 0. \tag{6}$$

Here α and β are the heat transfer coefficients and the ratio of the water-yielding capacity of a soil to the atmosphere, respectively, $T_a = T_a(t)$ and $\Omega_a = \Omega_a(t)$ are the temperature and the moisture of the atmosphere, they are assumed to be continue functions. Functions $T_0(z)$ and $\Omega_0(z)$ are continuous and they must satisfy the gluing conditions (5)-(6). Additionally we assume that these functions can be well approximated by quadratic functions on each interval $[z_{k-1}, z_k]$:

$$T_0(z) = \frac{1}{2}a_T^k(z - z_{k-1})^2 + b_T^k(z - z_{k-1}) + c_T^k, \tag{7}$$

$$\Omega_0(z) = \frac{1}{2}a_\Omega^k(z - z_{k-1})^2 + b_\Omega^k(z - z_{k-1}) + c_\Omega^k, \tag{8}$$

i.e., piecewise constant functions a_T, b_T, c_T, a_Ω, b_Ω and c_Ω with the values a_T^k, b_T^k, c_T^k, a_Ω^k, b_Ω^k and c_Ω^k in k-th interval are known.

Introduce the new functions:

$$\tau(z,t) = T(z,t) - T_0(z), \quad w(z,t) = \Omega(z,t) - \Omega_0(z) \tag{9}$$

These functions satisfy the relations:

$$
\begin{aligned}
C_0\frac{\partial\tau}{\partial t} - C_b\frac{\partial w}{\partial t} &= \frac{\partial}{\partial z}\left(\lambda\frac{\partial\tau}{\partial z}\right) + f_T,\\
\frac{\partial w}{\partial t} &= \frac{\partial}{\partial z}\left(\eta\frac{\partial w}{\partial z} + \mu\frac{\partial\tau}{\partial z}\right) + f_\Omega,
\end{aligned}
\tag{10}
$$

$$
\left.\left(\lambda\frac{\partial\tau}{\partial z} + \alpha\big(\tau - T_a\big)\right)\right|_{z=0} = \phi_T, \quad \tau|_{z=H} = \tau_H, \tag{11}
$$

$$
\left.\left(\eta\frac{\partial w}{\partial z} + \beta\big(w - \Omega_a\big)\right)\right|_{z=0} = \phi_\Omega, \quad w|_{z=H} = w_\Omega \tag{12}
$$

$$
\left[\lambda\frac{\partial\tau}{\partial z}\right]_{z_k} = 0, \quad [\tau]_{z_k} = 0, \tag{13}
$$

$$
\left[\eta\frac{\partial w}{\partial z} + \mu\frac{\partial\tau}{\partial z}\right]_{z_k} = 0, \quad [w]_{z_k} = 0, \tag{14}
$$

$$
\tau|_{t=0} = 0, \quad w|_{t=0} = 0, \tag{15}
$$

where the following notations are introduced

$$
\begin{aligned}
f_T &= \lambda a_T,\\
f_\Omega &= \eta a_\Omega + \mu a_T,\\
\phi_T &= -\lambda b_T^1 - \alpha c_T^1,\\
\phi_\Omega &= -\eta b_\Omega^1 - \beta c_\Omega^1,\\
\tau_H &= T_H - \frac{1}{2}a_T^N(H - z_{N-1})^2 - b_T^N(H - z_{N-1}) - c_T^N,\\
w_\Omega &= \Omega_H - \frac{1}{2}a_\Omega^N(H - z_{N-1})^2 - b_\Omega^N(H - z_{N-1}) - c_\Omega^N.
\end{aligned}
$$

Here f_T and f_Ω are piecewise constant functions, ϕ_T, ϕ_Ω, τ_H and w_Ω are constants.

The main goal of our work is to obtain analytical expressions for the functions $\hat\tau(z,p)$ and $\hat w(z,p)$ those are the images of the Laplace transforms of the functions $\tau(z,t)$ and $w(z,t)$. Here $p = \epsilon + i\omega$ is the Laplace transform parameter, ϵ is the attenuation parameter and ω is the circular time frequency.

We introduce the following notations:

$$
U = \begin{bmatrix} \hat\tau \\ \hat w \end{bmatrix}, \quad A = \begin{bmatrix} \lambda & 0 \\ \mu & \eta \end{bmatrix}, \quad D = p\begin{bmatrix} C_0 & -C_b \\ 0 & 1 \end{bmatrix}, \quad F = \frac{1}{p}\begin{bmatrix} f_T \\ f_\Omega \end{bmatrix},
$$

$$
A_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \eta \end{bmatrix}, \quad B_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad G_0 = \begin{bmatrix} \phi_T + \alpha\hat T_0(p) \\ \phi_\Omega + \beta\hat\Omega_0(p) \end{bmatrix}, \quad G_H = \begin{bmatrix} \tau_T \\ w_\Omega \end{bmatrix}
$$

where $\hat{T}_0(p)$ and $\hat{\Omega}_0(p)$ are the images of the Laplace transform for the functions $T_0(t)$ and $\Omega(t)$.

We apply the Laplace transform to the relations (10)-(15) and obtain

$$\frac{\partial}{\partial z}\left(A\frac{\partial U}{\partial z}\right) - DU = -F, \tag{16}$$

$$\left(A_0\frac{\partial U}{\partial z}x + B_0\right)\bigg|_{z=0} = G_0, \quad U|_{z=H} = G_H, \tag{17}$$

$$\left[A\frac{\partial U}{\partial z}\right]_{z_k} = 0, \ \ [U]_{z_k} = 0. \tag{18}$$

To obtain analytical expressions for the solution of (16)-(18) we use the idea of the layer stripping method. To this purpose we introduce the square matrix X and the vector Y by the following correlation:

$$A\frac{\partial}{\partial z}U = XU + Y. \tag{19}$$

Substituting (19) in (16)-(18), it is easy to obtain the statements for X and Y:

$$X' + XA^{-1}X = D, \quad X|_{z=0} = -AA_0^{-1}B_0, \quad [X]_{z_k} = 0, \tag{20}$$

$$Y' + XA^{-1}Y = -F, \quad Y|_{z=0} = AA_0^{-1}G_0, \quad [Y]_{z_k} = 0. \tag{21}$$

Let $z \in [z_{k-1}, z_k]$.

Now we want to obtain the solution of the differential matrix Riccati equation (20). Let R^k be the constant matrix and the partial solution of (20), i.e., it is a solution of the matrix Riccati equation

$$R^kA^{-1}R^k = D. \tag{22}$$

Here superscript k denotes that elements of the matrices R^k, A, and D corresponds to k-th layer. Also here and below the superscript k for matrices $\hat{C}$, $\check{C}$, A, and D have been omitted for simplicity.

Let $Z = X - R$ be satisfy the differential matrix Bernoulli equation

$$Z' + ZA^{-1}Z + Z\hat{C} + \check{C}Z = 0, \tag{23}$$

where

$$\hat{C} = A^{-1}R^k, \quad \check{C} = R^kA^{-1}, \quad \check{C}A\hat{C} = D. \tag{24}$$

Set $L = Z^{-1}$. This matrix satisfies the linear inhomogeneous differential matrix equation

$$L' = \hat{C}L + L\check{C} + A^{-1}. \tag{25}$$

Consequently, the solution of the differential matrix Riccati equation can be represented in the form

$$X(z) = R^k + L^{-1}(z), \quad z \in [z_{k-1}, z_k], \tag{26}$$

$$
\begin{aligned}
L(z) &= \mathrm{e}^{\hat{C}(z-z_{k-1})}(X^{k-1}-R^k)^{-1}\mathrm{e}^{\check{C}(z-z_{k-1})} \\
&+ \mathrm{e}^{\hat{C}(z-z_{k-1})}\int\limits_{z_{k-1}}^{z}\mathrm{e}^{-\hat{C}(s-z_{k-1})}A^{-1}\mathrm{e}^{-\check{C}(s-z_{k-1})}\mathrm{d}s\,\mathrm{e}^{\check{C}(z-z_{k-1})}.
\end{aligned} \tag{27}
$$

Here $X^{k-1} = X|_{z=z_{k-1}}$.

Now we obtain the solution of the differential equation (21). The solution of this equation can be expressed through the matrix Q, which is the solution of the problem

$$Q' + XA^{-1}Q = 0, \quad Q|_{z=z_{k-1}} = E. \tag{28}$$

Apparently, the matrix Q cannot be obtained from this equation in an analytical form. From (28) we obtain

$$Q' + ZA^{-1}Q + R^kA^{-1}Q = 0$$

and

$$-(Q^{-1})' + Q^{-1}ZA^{-1} + Q^{-1}\check{C} = 0.$$

We multiply this equation on the left by matrix Z and equation (23) on the right by matrix Q^{-1}, the summation gives the following result:

$$(Q^{-1}Z)' + (Q^{-1}Z)\hat{C} = 0$$

or

$$(LQ)' - \hat{C}(LQ) = 0.$$

Hence, the analytical expression for the matrix Q can be easily obtained

$$Q(z) = Z(z)\mathrm{e}^{\hat{C}(z-z_{k-1})}(X^{k-1}-R^k)^{-1}, \quad z \in [z_{k-1}, z_k].$$

Taking this into account, we obtain the solution of the equation (21) in the form

$$
\begin{aligned}
Y(z) &= Q(z)Y^{k-1} + Q(z)\int\limits_{z_{k-1}}^{z} Q^{-1}(s)F\,\mathrm{d}s \\
&= Z(z)\mathrm{e}^{\hat{C}(z-z_{k-1})}(X^{k-1}-R^k)^{-1} + Z(z)\mathrm{e}^{\hat{C}(z-z_{k-1})}\int\limits_{z_{k-1}}^{z}\mathrm{e}^{-\hat{C}(s-z_{k-1})}L(s)\mathrm{d}s\,F.
\end{aligned}
$$

In this expression, the integral can be simplified as follows:

$$
\begin{aligned}
\int\limits_{z_{k-1}}^{z}\mathrm{e}^{-\hat{C}(s-z_{k-1})}L(s)\mathrm{d}s &= \int\limits_{z_{k-1}}^{z}\mathrm{e}^{-\hat{C}(s-z_{k-1})}(L' - \hat{C}L - A^{-1})\mathrm{d}s\,\check{C}^{-1} \\
&= \int\limits_{z_{k-1}}^{z}\left(\mathrm{e}^{-\hat{C}(s-z_{k-1})}L\right)'\mathrm{d}s\,\check{C}^{-1} - \int\limits_{z_{k-1}}^{z}\mathrm{e}^{-\hat{C}(s-z_{k-1})}\mathrm{d}s\,A^{-1}\check{C}^{-1} \\
&= \left(\mathrm{e}^{-\hat{C}(s-z_{k-1})}L(s)\check{C}^{-1} + \mathrm{e}^{-\hat{C}(s-z_{k-1})}\hat{C}^{-1}A^{-1}\check{C}^{-1}\right)\Big|_{s=z_{k-1}}^{s=z}.
\end{aligned}
$$

After that, we obtain

$$Y(z)=\big(X(z)-R^k\big)\left(\mathrm{e}^{\hat{C}(z-z_{k-1})}M^k-D^{-1}F^k\right)-\check{C}^{-1}F^k,\quad z\in[z_{k-1},z_k],\tag{29}$$

where

$$M^k=\big(X^{k-1}-R^k\big)^{-1}\big(Y^{k-1}+\check{C}^{-1}F^k\big)+D^{-1}F^k,\quad Y^{k-1}=Y|_{z=z_{k-1}}.$$

Since the solutions of the differential equations (20) and (21) are known, we can obtain the solution of the differential equation (19). The solution of this equation is expressed through the matrix Q, which is the solution of the problem

$$Q'-A^{-1}XQ=0,\quad Q|_{z=z_k}=E.\tag{30}$$

From (30) we get

$$Q'-A^{-1}ZQ-\hat{C}Q=0$$

We multiply this equation on the left by matrix Z and equation (23) on the right by matrix Q, the summation gives the following result:

$$(ZQ)'-\check{C}(ZQ)=0.$$

Hence, the analytical expression for the matrix Q can be easily obtained

$$Q(z)=L(z)\mathrm{e}^{\check{C}(z_k-z}(X^{k-1}-R^k),\quad z\in[z_{k-1},z_k].\tag{31}$$

Taking (31) and (29) into account, we can write the solution of the equation (19):

$$\begin{aligned}
U(z) &= Q(z)U^k+Q(z)\int\limits_{z_{k-1}}^{z}Q^{-1}(s)Y(s)\,\mathrm{d}s\\
&= L(z)\mathrm{e}^{\check{C}(z_k-z)}(X^{k-1}-R^k)U^k+L(z)\mathrm{e}^{\check{C}(z_k-z)}\int\limits_{z_k}^{z}\mathrm{e}^{-\check{C}(z_k-s)}Z(s)A^{-1}Y(s)\,\mathrm{d}s\\
&= L(z)\mathrm{e}^{\check{C}(z_k-z)}(X^{k-1}-R^k)U^k\\
&+ L(z)\mathrm{e}^{\check{C}(z_k-z)}\int\limits_{z_k}^{z}\mathrm{e}^{-\check{C}(z_k-s)}Z(s)A^{-1}Z(s)\mathrm{e}^{\hat{C}(s-z_{k-1})}\mathrm{d}s\,M^k\\
&- L(z)\mathrm{e}^{\check{C}(z_k-z)}\int\limits_{z_k}^{z}\mathrm{e}^{-\check{C}(z_k-s)}\big(Z(s)A^{-1}Z(s)+Z\hat{C}\big)\,\mathrm{d}s\,\hat{C}^{-1}A^{-1}\check{C}^{-1}F^k.
\end{aligned}$$

First integral can be simplified as follows:

$$\begin{aligned}
\int\limits_{z_k}^{z}\mathrm{e}^{-\check{C}(z_k-s)}Z(s)A^{-1}Z(s)\mathrm{e}^{\hat{C}(s-z_{k-1})}\mathrm{d}s &= -\int\limits_{z_k}^{z}\mathrm{e}^{-\check{C}(z_k-s)}\big(Z'+\check{C}Z+Z\hat{C}\big)\mathrm{e}^{\hat{C}(s-z_{k-1})}\mathrm{d}s\\
&= -\int\limits_{z_k}^{z}\left(\mathrm{e}^{-\check{C}(z_k-s)}Z(s)\mathrm{e}^{\hat{C}(s-z_{k-1})}\right)'\mathrm{d}s\\
&= -\left(\mathrm{e}^{-\check{C}(z_k-s)}Z(s)\mathrm{e}^{\hat{C}(s-z_{k-1})}\right)\Bigg|_{s=z_k}^{s=z}.
\end{aligned}$$

Second integral can be simplified as follows:

$$\begin{aligned}
\int\limits_{z_k}^{z} \mathrm{e}^{-\check{C}(z_k-s)}\big(Z(s)A^{-1}Z(s)+Z\hat{C}\big)\,\mathrm{d}s &= -\int\limits_{z_k}^{z} \mathrm{e}^{-\check{C}(z_k-s)}\big(Z'+\check{C}Z\big)\mathrm{d}s \\
&= -\int\limits_{z_k}^{z} \Big(\mathrm{e}^{-\check{C}(z_k-s)}Z(s)\Big)'\,\mathrm{d}s \\
&= -\Big(\mathrm{e}^{-\check{C}(z_k-s)}Z(s)\Big)\Big|_{s=z_k}^{s=z}.
\end{aligned}$$

Taking these into account, finally we get the expression for $U(z)$

$$U(z)=L(z)\mathrm{e}^{\check{C}(z_k-z)}V^k-\mathrm{e}^{\hat{C}(z-z_{k-1})}M^k+D^{-1}F^k, \quad z\in[z_{k-1},z_k], \tag{32}$$

where

$$V^k=\big(X^{k-1}-R^k\big)\Big(U^k+\mathrm{e}^{\hat{C}h_k}M^k-D^{-1}F^k\Big).$$

The matrices R^k, $\hat{C}$, and $\check{C}$ are included in all expressions above. They are connected by the relations (24). From (22) the matrix $\hat{C}$ is the solution of the matrix equation

$$A\hat{C}^2=D. \tag{33}$$

It is well known (see, for example, [28]), if a matrix is the solution of the matrix equation (33), then its eigenvalues must satisfy the characteristic equation

$$|A\xi^2-D|=0 \quad \Rightarrow \quad \lambda\eta\xi^4-p(C_0\eta+\lambda+\mu C_b)\xi^2+p^2C_0=0 \tag{34}$$

(it should be noted that (34) is the characteristic equation for the system (16) too). Four roots of this equation can be calculated from the equality

$$\xi^2=p\frac{(C_0\eta+\lambda+\mu C_b)\pm\sqrt{(C_0\eta+\lambda+\mu C_b)^2-4C_0}}{\lambda\eta}. \tag{35}$$

By physical reasons, the value under the root is non-negative, hence, two roots have positive real part and two roots have negative real part.

To obtain the matrix $\hat{C}$ (see [28]), we need to solve the characteristic equation (34), to take any pair of roots, and to construct the Jordan form J of the matrix $\hat{C}$; the transition matrix can be found by substitution of $\hat{C}=T^{-1}JT$ in the matrix equation (33). From the point of computing view, this method is unsatisfactory. A simple way to calculate the matrix $\hat{C}$ is proposed in the works [13, 14, 20], if the eigenvalues of $\hat{C}$ are known. Let ξ_1 and ξ_2 be known and $\xi_1+\xi_2\neq 0$. Each matrix is a solution of its characteristic equation, i.e., the matrix equation holds

$$\hat{C}^2-(\xi_1+\xi_2)\hat{C}+\xi_1\xi_2E=0.$$

From this matrix equation and matrix equations (33) we easily obtain the following equality

$$\hat{C}=\frac{1}{\xi_1+\xi_2}(A^{-1}D+\xi_1\xi_2E). \tag{36}$$

Using the matrix $\hat{C}$ and the relations (24), we can compute the matrix R^k and $\check{C}$.

It should be noted that the matrix $\check{C}$ satisfies the matrix equation $\check{C}^2 A = D$ for which the characteristic equation has the form (34), it means that the eigenvalues of $\hat{C}$ and $\check{C}$ are equal. For the matrix $\check{C}$ the expression similar to (36) can be obtained.

For stable computing matrix exponentials, we use the idea from [29]. In our case, we have the following correlations:

$$\mathrm{e}^{\hat{C}z} = E\psi(z,\xi_1) + (\hat{C} - \xi_1 E)\psi(z,\xi_1,\xi_2),$$

where

$$\begin{cases} \psi(z,\xi_1) = \mathrm{e}^{\xi_1 z}, \qquad \psi(z,\xi_1,\xi_2) = \dfrac{1}{\xi_1 - \xi_2}(\mathrm{e}^{\xi_1 z} - \mathrm{e}^{\xi_2 z}), \qquad \text{если} \;\; \xi_1 \neq \xi_2, \\ \psi(z,\xi_1) = \mathrm{e}^{\xi_1 z}, \qquad \psi(z,\xi_1,\xi_1) = z\mathrm{e}^{\xi_1 z}, \qquad \text{если} \;\; \xi_1 = \xi_2. \end{cases}$$

For the correlation (27) for the matrix $L(z)$, we give a way to compute the second member

$$\Xi(z) = \mathrm{e}^{\hat{C}(z-z_{k-1})} \int\limits_{z_{k-1}}^{z} \mathrm{e}^{-\hat{C}(s-z_{k-1})} A^{-1} \mathrm{e}^{-\check{C}(s-z_{k-1})} \mathrm{d}s \, \mathrm{e}^{\check{C}(z-z_{k-1})}.$$

It is easy to see that the matrix $\Xi(z)$ satisfies the matrix equation

$$\hat{C}\Xi + \Xi\check{C} = \mathrm{e}^{\hat{C}(z-z_{k-1})} A^{-1} \mathrm{e}^{\check{C}(z-z_{k-1})} - A^{-1} \equiv \tilde{C}, \tag{37}$$

which is equivalent to the problem for the matrix and vectors of fourth order:

$$\begin{bmatrix} \hat{C} + \check{C}_{11}E & \check{C}_{21}E \\ \check{C}_{12}E & \hat{C} + \check{C}_{22}E \end{bmatrix} \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix} = \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix},$$

where Ξ_j and $\tilde{C}_j$ are j-th column of the matrices Ξ and $\tilde{C}$, $\check{C}_{nm}$ are elements of the matrix $\check{C}$.

Since the eigenvalues of the matrices $\hat{C}$ and $\check{C}$ are equal, then the equation (37) has a unique solution for any right part $\tilde{C}$.

To compute the column Ξ_j ($j = 1, 2$) of the matrix $\Xi(z)$, we obtain the following relations:

$$\begin{aligned} \Xi_1 &= \left[(\hat{C} + \check{C}_{22}E)(\hat{C} + \check{C}_{11}E) - \check{C}_{12}\check{C}_{21}E\right]^{-1} \left[(\hat{C} + \check{C}_{22}E)\tilde{C}_1 - \check{C}_{21}\tilde{C}_2\right], \\ \Xi_2 &= \left[(\hat{C} + \check{C}_{11}E)(\hat{C} + \check{C}_{22}E) - \check{C}_{12}\check{C}_{21}E\right]^{-1} \left[(\hat{C} + \check{C}_{11}E)\tilde{C}_2 - \check{C}_{12}\tilde{C}_1\right]. \end{aligned} \tag{38}$$

Now we demonstate how to compute a_T^k, b_T^k, and c_T^k ($k = \overline{1,N}$) for (7). Let the function $T_0(z)$ be taken. Since the values $T_0(z_k)$ ($k = \overline{0,N}$) are known then

$$c_T^k = T_0(z_{k-1}), \quad k = \overline{1,N}. \tag{39}$$

At the point $z = 0$ the first boundary condition (3) must be satisfied, hence we have

$$\lambda_1 b_T^1 + \alpha(T_0(0) - T_a(0)) = 0. \tag{40}$$

At the points z_k ($k=\overline{1,N-1}$) the gluing conditions (5) give us the following correlations

$$\begin{aligned} \frac{1}{2}a_T^k h_k^2 + b_T^k h_k + T_0(z_{k-1}) &= T_0(z_k), \\ \lambda_k(a_T^k h_k + b_T^k) &= \lambda_{k+1} b_T^{k+1}. \end{aligned} \tag{41}$$

At the point $z=z_N$ the second boundary condition (3) must be satisfied, hence we have

$$\frac{1}{2}a_T^N h_N^2 + b_T^N h_N + T_0(z_{N-1}) = T_0(z_N). \tag{42}$$

From (40) and (41) we get the recurrence relations

$$\begin{aligned} b_T^{k+1} &= \frac{\lambda_k}{\lambda_{k+1}}\left(2\frac{T_0(z_k)-T_0(z_{k-1})}{h_k} - b_T^k\right), \quad k=\overline{1,N-1}, \\ b_T^1 &= \frac{\alpha}{\lambda_1}\Big(T_a(0)-T_0(0)\Big). \end{aligned} \tag{43}$$

From (41) and (42) we obtain

$$a_T^k = \frac{2}{h_k}\left(\frac{T_0(z_N)-T_0(z_{N-1})}{h_k} - b_T^k\right), \quad k=\overline{1,N}. \tag{44}$$

Similarly, the expressions for a_Ω^k, b_Ω^k, and c_Ω^k ($k=\overline{1,N}$) can be obtained from (8) as follows:

$$\begin{aligned} c_\Omega^k &= \Omega_0(z_{k-1}), \quad k=\overline{1,N}, \\ b_\Omega^{k+1} &= \frac{\eta_k}{\eta_{k+1}}\left(2\frac{\Omega_0(z_k)-\Omega_0(z_{k-1})}{h_k} - b_\Omega^k + r^{k+1}\right), \quad k=\overline{1,N-1}, \\ & r^{k+1} = \mu_{k+1}b_T^{k+1} - \mu_k\big(h_k a_T^k h_k + b_T^k\big) \\ b_T^1 &= \frac{\alpha}{\lambda_1}\Big(\Omega_a(0)-\Omega_0(0)\Big), \\ a_\Omega^k &= \frac{2}{h_k}\left(\frac{\Omega_0(z_N)-\Omega_0(z_{N-1})}{h_k} - b_\Omega^k\right), \quad k=\overline{1,N}. \end{aligned} \tag{45}$$

All necessary analytical expressions to compute $U(z)$ are obtained. In the next section we give a procedure for computing $U(z)$ at any point z in any interval $[z_{m-1}, z_m]$.

3 Order of operations for calculating $U(z)$

Now we give the procedure of layer-stripping method to compute the solution of the problem (16)-(18). Let we need to know $U(z)$ for $z \in [z_{m-1}, z_m]$.

- Values a_T^k, b_T^k, c_T^k, a_Ω^k, b_Ω^k, and c_Ω^k ($k=\overline{1,N}$) can be calculated using relations (39), (43), (44), and (45).

- In each interval $[z_{k-1}, z_k]$ $(k = \overline{1, N})$ it is necessary
  - to solve the characteristic equation (34), to choose two roots with negative real parts;
  - to calculate the matrices $\hat{C}$, $\check{C}$, and R^k (see (36) and the connection (24));
  - to calculate the matrix X^k and the vector Y^k using the following recurrent correlation:

$$
\begin{aligned}
X^k &= R^k + (L^k)^{-1}, \\
L^k &= \mathrm{e}^{\hat{C}h_k}(X^{k-1} - R^k)^{-1}\mathrm{e}^{\check{C}h_k} + \mathrm{e}^{\hat{C}h_k}\int\limits_{z_{k-1}}^{z_k} \mathrm{e}^{-\hat{C}(s-z_{k-1})}A^{-1}\mathrm{e}^{-\check{C}(s-z_{k-1})}\mathrm{d}s\,\mathrm{e}^{\check{C}h_k}, \\
Y^k &= (X^k - R^k)\left(\mathrm{e}^{\hat{C}h_k}M^k - D^{-1}F^k\right) - \check{C}^{-1}F^k, \\
X^0 &= -AA_0^{-1}B_0, \\
Y^0 &= AA_0^{-1}G_0,
\end{aligned}
$$

  (see (26)-(27) and (29)); for L^k the second member can be calculated using (38) where

$$\tilde{C} = \mathrm{e}^{\hat{C}h_k}A^{-1}\mathrm{e}^{\check{C}h_k} - A^{-1}.$$

- $U^N = G_H$ (see (17)).
- The vector U^k $(k = \overline{N, m+1})$ should be calculated using the following recurrent corellation:

$$U^{k-1} = L^{k-1}\mathrm{e}^{\check{C}h_k}V^k - M^k + D^{-1}F^k$$

  (see (32)).
- From (32) the vector $U(z)$ can be calculated for $z \in [z_{m-1}, z_m]$.

Thus, the algorithm for calculating the vector U(z) is completely described.

To solve inverse problems, as a rule, we need to know the vector U on a surface, i.e., U^0. In this case we should compute U^k, using the recurrence relation for $k = \overline{N, 1}$.

It should be noted that we chose the eigenvalues of the matrices $\hat{C}$ and $\check{C}$ with negative real parts. In this case, the rounding error will not be accumulated in recurrent calculations, because all expressions with matrix exponents are computed stably.

The second important property of the algorithm is: for its implementation on each interval $[z_{k-1}, z_k]$, all actions is reduced to an addition, multiplication and inversion for square matrices of the second order.

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References

[1] Tikhonov A.N., Shakhsuvarov D.N., *Method for calculation of electromagnetic fields, excited by alternating current in layered media*, Izvestiya AN SSSR, ser. Geofizicheskaya, 3 (1956), 251-254. (In Russian)

[2] Dmitriev V.I., *A general method for calculating electromagnetic field in a stratified medium*, Numerical methods and programming, 10 (1968), 55-65. (In Russian)

[3] Dmitriev V.I., Fedorova E.A., *Numerical studies of electromagnetic fields in layered media*, Numerical methods and programming, 32 (1980), 150-183. (In Russian)

[4] Akkuratov G.V., Dmitriev V.I., *Calculation method for field of the steady-state elastic waves in layered media.* In the book: Numerical method in Geophysics. Moskva, MGU, 1979, 3-12. (In Russian)

[5] Akkuratov G.V., Dmitriev V.I., *Calculation method for field of the steady-state elastic waves in layered media*, Journal of Computational mathematics and mathematical physics, 24. 2 (1984), 272-286. (In Russian)

[6] Fat'yanov A.G., Mikhailenko B.G., *Method of calculating non-stationary wave fields in layered elastic inhomogeneous media*, Doklady AN SSSR, 301. 4 (1988), 834-839.

[7] Fat'yanov A.G., *Semi-analytical method for solving direct dynamic problems in layered media*, Doklady AN SSSR, 1990, 310. 2 (1990), 323-327.

[8] Fat'yanov A.G., *Non-stationary seismic wave field in heterogeneous environments anizotropnykh with absorption of energy.* Novosibirsk, 1989, Preprint of Coputing Center SO AN, № 857. (In Russian)

[9] Karchevsky A.L., *Numerical Solution to the One-Dimensional Inverse Problem for an Elastic System*, Transactions (Doklady) of the Russian Academy of Sciences/Earth Science Section, 375. 8 (2000), 1325-1328.

[10] Karchevsky A.L., Fatianov A.G., *Numerical solution of the inverse problem for a system of elasticity with the aftereffect for a vertically inhomogeneous medium*, Sib. Zh. Vychisl. Mat., 4. 3 (2001), 259–268.

[11] Pavlov V.M., *A convenient technique for calculating synthetic seismograms in layered half-space*, Proceedings of the Int. Conf. "Problems of Geocosmos", St. Peteburg, 03-08 June 2002, 320-323.

[12] Kurpinar E., Karchevsky A.L., *Numerical solution of the inverse problem for the elasticity system for horizontally stratified media*, Inverse Problems, 20. 3 (2004), 953-976.

[13] Karchevsky A.L., *A numerical solution to a system of elasticity equations for layered anisotropic media*, Russian Geology and Geophysics, 46. 3 (2005), 339-351.

[14] Karchevsky, A.L., *The direct dynamical problem of seismics for horizontally stratified media*, Siberian Electronic Mathematical Reports, 2. (2005), 23-61. (http://semr.math.nsc.ru/V2/p23-61.pdf)

[15] Karchevsky A.L., *Numerical reconstruction of medium parameters of member of thin anisotropic layers*, J. Inv. Ill-Posed Problems, 12. 5 (2004), 519-634.

[16] Karchevsky A.L., *Analysis of solving of the inverse dynamical problem of seismics for horizontally stratified anisotropic media*, Russian Geology and Geophysics, 47. 11 (2006), 1150-1164.

[17] Karchevsky, A.L., *Algorithm for reconstruction of the elastic constants of an anisotropic layer lying in an isotropic horizontally stratified medium*, Siberian Electronic Mathematical Reports, 4. (2007), 20-51. (http://semr.math.nsc.ru/v4/p20-51.pdf)

[18] Kurpinar E., Karchevsky A.L., *Finding of the elastic parameters of a horizontal (thinly stratified) anisotropic layer*, Applicable Analysis, 87. 10 & 11 (2008), 1179-1212.

[19] Tabarovsky L.A., Epov M.I., *The electromagnetic field of harmonic sources in layered anisotropic media*, Geplpgia i Geofizika, 1. (1977), 101-109.

[20] Karchevsky A.L., *A frequency-domain analytical solution of Maxwell's equations for layered anisotropic media*, Russian Geology and Geophysics, 48. 8 (2007), 689-695. (In Russian)

[21] Plotkin V.V., *Inversion of heterogeneous anisotropic magnetotelluric responses*, Russian Geology and Geophysics, 53. 8 (2012), 829-836.

[22] Plotkin V.V., *The restoration of tensor conductivity on borehole data*, Interexpo Geo-Siberia, 1. 2 (2012), 28-32. (In Russian)

[23] Rysbaiuly B., Baimankulov A.T., *Stability of difference schemes of the direct and conjugate problem for the determination of the coefficient diffusions of soil water*, Doklady NAN RK, 3 (2009), 5-8. (In Russian)

[24] Rysbaiuly B., Baimankulov A.T., *Variational-difference method for determining the diffusion coefficient of soil water*, International Journal of Academic Research, 5 (2010), 84-91.

[25] Martynov G.A., *Heat and moisture transport in frozen and thawed soils*. In the book: Fundamentals of Geocryology (Permafrost studies). Ed. Tsytovich N.A., Moskva, 1959, 153-192. (In Russian)

[26] Globus A.M., *Physics of non-isothermal inside-soil moisture exchange*, Leningrad, Gidrometizdat, 1983. (In Russian)

[27] Chistotinov L.V., *Moisture migration in freezing non-water-saturated soils*, Moskva, Nauka, 1973. (In Russian)

[28] Gantmakher F.R., *Marix thiory*, Moskva, Nauka, 1988. (In Russian)

[29] Godunov S.K., *Exponent matrix, the Green's matrix and the Lopatinsky condition*. Novosibirsk, NGU, 1983. (In Russian)

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